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Invariant eigendistributions on $SU_{p,q}$

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Abstract

The theory of global characters of semisimple Lie groups as invariant eigendistributions, proposed by Harish-Chandra, is used to determine explicitly the global characters of $SU_{p,q}$. It is shown that they are invariant, tempered eigendistributions on $SU_{p,q}$. The adjoint invariant distributions on $SU_{p,q}$ are studied in detail. For the special case of $SU_{1,1}$, these global characters reduce to the results already obtained for $SU_{1,1}$. The paper contains several new results pertaining to the group $SU_{p,q}$ and these results are explicitly used. © 2002 Published by Elsevier Science B.V.

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1. Introduction

Let $(\rho, \mathcal{H})$ denote an irreducible unitary representation of a real semisimple Lie group G over a Hilbert space $\mathcal{H}$. Let $\mathfrak{g}$ be the Lie algebra of G. Let K be a maximal compact subgroup of G and $\mathfrak{k}$ be its corresponding Lie algebra. Let $\mathfrak{g}_{\mathbf{c}}$ be the complexification of $\mathfrak{g}$. Then from the representation $(\rho, \mathcal{H})$, one obtains the $(\mathfrak{g}_{\mathbf{c}}, K)$-modules, which correspond to two representations: one of $\mathfrak{g}$ and the other of K, the latter being the restriction of the algebra representation of $\mathfrak{g}$ to $\mathfrak{k}$. One of Harish-Chandra's principal results [6] was that the space $\mathcal{H}_k$ of K-finite vectors in an irreducible unitary representation $(\rho, \mathcal{H})$ is an irreducible $(\mathfrak{g}_{\mathbf{c}}, K)$-module, and that the mapping $\rho \to \mathcal{H}_k$ induces a bijection of the unitary dual $\hat{G}$ with the set of isomorphism classes of irreducible unitary $(\mathfrak{g}_{\mathbf{c}}, K)$-modules. Furthermore,

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he proved [7] that any irreducible $(\mathfrak{g}_{\mathbf{c}}, K)$-module is of the form $\mathcal{H}_k$ for some irreducible representation $(\rho, \mathcal{H})$ which has an infinitesimal character, showing that the formulation of an arbitrary irreducible representation of G is built into the notion of an irreducible $(\mathfrak{g}_{\mathbf{c}}, K)$-module.

An irreducible unitary representation $(\rho, \mathcal{H})$ of a locally compact group G is said to be square integrable on G if it has a non-zero, square integrable matrix coefficient (matrix element), given by the function $x \mapsto (\rho_x u, v)$, $x \in G$, $u, v \in \mathcal{H}$. Following Harish-Chandra, one defines for a representation $(\rho, \mathcal{H})$ of a semisimple Lie group G the operator $\rho(f) = \int_G f(x)\rho_x \, \mathrm{d}\mu(x)$, $f \in C_{\mathrm{c}}^{\infty}(G)$, which is of trace class. Furthermore, the linear functional $T_\omega : f \mapsto \mathrm{tr}(\rho(f))$ is a distribution on G, where ω is the equivalence class containing the representation $(\rho, \mathcal{H})$. T_ω is called the *character* of the representations in ω and it determines ω completely. Also, T_ω is an invariant distribution, i.e., a distribution that is invariant under all inner automorphisms of G. If $\{u_i\}_{i\geq 1}$ is an orthonormal basis of $\mathcal{H}$, then for $x \in G$ and $f \in C_{\mathrm{c}}^{\infty}(G)$, $T_\omega(f) = \sum_{i\geq 1}\int_G (\rho_x u_i, u_i) f(x) \, \mathrm{d}\mu(x)$.

Let $\mathfrak{Z}$ denote the center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ [2,5], and let $\chi_\omega \in \mathrm{Hom}(\mathfrak{Z}, \mathbb{C})$ denote the infinitesimal character of ω. Then for $\mathfrak{z} \in \mathfrak{Z}$ and $x \in G$ one has the equation $\mathfrak{z}(\rho_x u_i, u_i) = \chi_\omega(\mathfrak{z})(\rho(x)u_i, u_i)$, $i \geq 1$ and hence for $f \in C_{\mathrm{c}}^{\infty}(G)$, $\mathfrak{z}T_\omega(f) = \chi_\omega(\mathfrak{z})T_\omega(f)$. Thus, T_ω is an *invariant eigendistribution* on G, and $\chi_\omega(\mathfrak{z})$ is the eigenvalue of $\mathfrak{Z}$ [8,10].

Furthermore, Harish-Chandra showed that distinct series of representations induced by a complete system of mutually non-conjugate Cartan subgroups [25] would fall into distinct classes in the unitary dual $\hat{G}$. He further showed [11,12] that G has a discrete series if and only if it has a compact Cartan subgroup and that the latter exists if and only if the rank of G is equal to the rank of K. Also, Harish-Chandra established that the representations induced by a Cartan subgroup H can be parameterized by the parameters of the characters of H', the set of all regular elements of H, and he determined these characters as invariant eigendistributions. In particular, for a compact Cartan subgroup, he proved that the corresponding discrete series of representations are square integrable and that this series of representations is complete.

The discrete and continuous series of irreducible unitary representations of $SU_{p,q}$, associated with the $(\min\{p, q\} + 1)$ non-conjugate Cartan subgroups, have been explicitly obtained in [27,28,31]. In [4], a class of degenerate representations of $SU_{p,q}$ and the trace of those representations are obtained. The invariant eigendistributions of Laplace operators of $SU_{p,q}$ associated with discrete series and an expression for the Plancherel formula for $SU_{p,q}$ have been obtained in [19–22]. The group $SU_{p,q}$ has one discrete series of representations, associated with the compact Cartan subgroup, and $\min\{p, q\}$ number of continuous series of representations, associated with the $\min\{p, q\}$ non-compact Cartan subgroups.

The organization of this paper is as follows. In Section 2, basic algebraic structures of $SU_{p,q}$ are discussed and relevant notations are established. By the *Kostant–Sugiura theorem* [25,29], there exist $(\min\{p, q\} + 1)$ non-conjugate Cartan subgroups for $SU_{p,q}$; these subgroups are obtained explicitly in Section 3 through root structures of $SU_{p,q}$. In Section 4, the characters are introduced and various properties of Harish-Chandra's *density functions* Δ, some of which are new, have been derived. In Sections 5 and 6, some basic and relevant properties of the universal enveloping algebra of $\mathfrak{sl}(n, \mathbb{C})$ are discussed, and its relation to the symmetric algebra of differential operators is established. The *Harish-Chandra homomorphism* γ, related to each Cartan subalgebra, is obtained in Section 7, and after introducing representation parameters in Section 8, a representation of the algebra of $SU_{p,q}$ on the *Gårding* domain is defined, and some of its basic properties are given in Section 9. The *global characters* of $SU_{p,q}$ associated with each Cartan subgroup H_j, $0 \le j \le \min\{p, q\}$, are explicitly obtained in Sections 10 and 11. Theorem 11.1 gives the main results. Our results agree with the case for $j = 0$ obtained in [4,19–21], and for $SU_{1,1}$ obtained in [14], the latter is found to be useful in scattering theory. It is then shown in Section 12 that these global characters are the invariant eigendistributions as defined by Harish-Chandra, and that these eigendistributions are tempered. In Section 13, the invariant eigendistributions of the contragradiant representations of $SU_{p,q}$ are obtained, and the result is given in Theorem 13.1. Finally, the adjoint invariant eigendistributions are discussed in Section 14, and some of their properties are proved. We have given here and there some materials, which are somewhat expository in nature, in order to establish notations and definitions and to provide the basic concepts required to prove and to foster understanding of the main results.

2. Basic algebraic structures and notation

Let $SU_{p,q}$, $p + q = n$, denote the pseudounitary, unimodular Lie group defined in matrix realization by $SU_{p,q} = \{g \in M_{n\times n}(\mathbb{C}) : \det(g) = 1,\ g^*\mathcal{J}_0 g = \mathcal{J}_0\}$, where $*$ denotes conjugate transpose and $M_{n\times n}(\mathbb{C})$, homeomorphic to $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$, denotes a matrix manifold of $(n \times n)$ matrices over $\mathbb{C}$. For definiteness, we take $p \le q$ throughout. The *metric operator* $\mathcal{J}_0$ is given by

$$\mathcal{J}_0 = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$$

with I_m an $(m \times m)$ unit matrix. The matrix operator $\mathcal{J}_0$ is bounded, self-adjoint ($\mathcal{J}_0^* = \mathcal{J}_0$), involutory ($\mathcal{J}_0^2 = I_n$), and unitary ($\mathcal{J}_0^* = \mathcal{J}_0^{-1}$). The group $SU_{p,q}$ is a non-compact, simple Lie group which leaves the Hermitian inner product $\langle u, \mathcal{J}_0 u\rangle = \sum_{n=1}^{p} \bar{u}_k u_k - \sum_{k=1}^{q} \bar{u}_{p+k} u_{p+k}$ $\forall u \in \mathcal{H}$ invariant, where $\mathcal{H}$ is an n-dimensional Hilbert space over $\mathbb{C}$. That is, one has for all $g \in SU_{p,q}$ and for all $u \in \mathcal{H}$, $\langle u, \mathcal{J}_0 u\rangle = \langle gu, \mathcal{J}_0 gu\rangle$. For the orthonormal basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of $\mathcal{H}$, where $\mathbf{e}_k$ is the n-tuple $(0, \dots, 0, 1_k, 0, \dots, 0)$, the inner product satisfies $\langle \mathbf{e}_j, \mathcal{J}_0 \mathbf{e}_k\rangle$ equals 0 if $j \ne k$, equals 1 if $1 \le j = k \le p$, and equals -1 if $p + 1 \le j = k \le n = p + q$. The center of $SU_{p,q}$ is given by $\mathcal{Z}(SU_{p,q}) = \{\mathrm{e}^{\iota(2\pi/n)m} I_n : m = 0, 1, \dots, n-1,\ \iota = \sqrt{-1}\}$. Hence $SU_{p,q}$ is connected, but not simply connected. Furthermore, the group $SU_{p,q}$ is reductive since it is closed under conjugate transpose.

It is possible to consider different matrix realizations of $SU_{p,q}$ by redefining the metric operator $\mathcal{J}_0$. For example, one may define the metric operators $\mathcal{J}_k$, $0 \le k \le p$, and obtain matrix realizations of $SU_{p,q}$ as $SU^k_{p,q} = \{g \in M_{n\times n}(\mathbb{C}) : \det(g) = 1,\ g^*\mathcal{J}_k g = \mathcal{J}_k\}$, where

$$\mathcal{J}_k = \begin{pmatrix} 0 & 0 & \hat{I}_k \\ 0 & \begin{pmatrix} I_{p-k} & 0 \\ 0 & -I_{q-k} \end{pmatrix} & 0 \\ \hat{I}_k & 0 & 0 \end{pmatrix}$$

with the $(k \times k)$ matrix $\hat{I}_k = (i_{mn})$ defined by $i_{mn} = 1$ if $m + n = k + 1$ and $= 0$ otherwise. There exists orthogonal transformations $\mathcal{O}_k,\ 0 \le k \le p$, which are elements of the orthogonal group O_{p+q}, under which the metric operator $\mathcal{J}_0$ transforms as $\mathcal{O}_k^{-1}\mathcal{J}_0\mathcal{O}_k = \mathcal{J}_k$ and for which $[\mathcal{O}_0, \mathcal{J}_0] = 0$. A matrix realization of $\mathcal{O}_k$ is given explicitly in [27]. Now, for every $g_{(k)} \in SU^k_{p,q}$ one has $\mathcal{O}_k g_{(k)} \mathcal{O}_k^* = g_0 \in SU^0_{p,q} = SU_{p,q}$. The group $SU^k_{p,q}$, for each k, acts on the Hilbert space $\mathcal{H}^k$, leaving the Hermitian inner product $\langle u, \mathcal{J}_k u\rangle,\ u \in \mathcal{H}^k$, invariant. That is, for all $g \in SU^k_{p,q}$, one has $\langle gu, \mathcal{J}_k gu\rangle = \langle u, \mathcal{J}_k u\rangle,\ 1 \le k \le p$. In what follows the realization of $SU_{p,q}$ will correspond to the realization of $SU^0_{p,q}$.

The group $SU_{p,q}$ is a subgroup of the special linear group $SL_n(\mathbb{C}) = \{g \in M_{n\times n}(\mathbb{C}) : \det(g) = 1\}$, which is a subgroup of the general linear group $GL_n(\mathbb{C}) = \{g \in M_{n\times n}(\mathbb{C}) : \det(g) \neq 0\}$. The Lie algebras of $SL_n(\mathbb{C})$ and $GL_n(\mathbb{C})$ are denoted, respectively, by $\mathfrak{s}l_n(\mathbb{C})$ and $\mathfrak{g}l_n(\mathbb{C})$. Specifically, $\mathfrak{s}l_n(\mathbb{C})$ is defined by $\mathfrak{s}l_n(\mathbb{C}) = \{X \in M_{n\times n}(\mathbb{C}) : \mathrm{tr}(X) = 0\}$. The Lie algebra of $SU_{p,q}$, denoted by $\mathfrak{s}u_{p,q}$, is defined by

$$\begin{aligned} \mathfrak{s}u_{p,q} &= \left\{ X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \in M_{n\times n}(\mathbb{C}) : X_{jj}^* = -X_{jj}, X_{12}^* = X_{21}, \mathrm{tr}(X) = 0 \right\} \\ &= \{X \in \mathfrak{s}l_n(\mathbb{C}) : \mathcal{J}_0 X^* \mathcal{J}_0 = -X\}, \end{aligned}$$

where X_{11} is a $(p \times p)$ matrix.

The Lie algebra $\mathfrak{s}u_{p,q}$ is a non-compact real form of $\mathfrak{s}l_n(\mathbb{C})$. Specifically, if $Z \in \mathfrak{s}l_n(\mathbb{C})$, then $(Z - \mathcal{J}_0 Z^* \mathcal{J}_0)$ and $(-\iota Z - \iota \mathcal{J}_0 Z^* \mathcal{J}_0),\ \iota = \sqrt{-1}$, are elements of $\mathfrak{s}u_{p,q}$, i.e., $Z = \frac{1}{2}(Z - \mathcal{J}_0 Z^* \mathcal{J}_0) + \iota\frac{1}{2}(-\iota Z - \iota\mathcal{J}_0 Z^*\mathcal{J}_0)$, and $\mathfrak{s}l_n(\mathbb{C}) = \{Z = X + \iota Y : X, Y \in \mathfrak{s}u_{p,q}\}$ and is the complexification of $\mathfrak{s}u_{p,q}$ [16]. In general, any realization of $\mathfrak{s}l_n(\mathbb{C})$ is isomorphic to the complexification of $\mathfrak{s}u_{p,q}$.

Let θ be an involutive mapping of $\mathfrak{s}l_n(\mathbb{C})$ onto itself, defined by $\theta(X + \iota Y) = -(X^* - \iota Y^*)\,\forall X, Y \in \mathfrak{s}u_{p,q}$. The action of θ on $\mathfrak{s}u_{p,q}$ is given by

$$\theta(X) = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} X \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \quad \forall X \in \mathfrak{s}u_{p,q}.$$

The $+1$ eigenspace of θ is given by the subalgebra

$$\mathfrak{k} = \left\{ X = \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} \in \mathfrak{s}u_{p,q} \right\} = \mathfrak{s}(\mathfrak{u}_p \oplus \mathfrak{u}_q),$$

and the -1 eigenspace of θ is given by the vector space

$$\mathfrak{p} = \left\{ X = \begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix} \in \mathfrak{s}u_{p,q} \right\}.$$

Thus θ induces the decomposition of $\mathfrak{su}_{p,q}$ given by $\mathfrak{su}_{p,q} = \mathfrak{k} \oplus \mathfrak{p}$. The subalgebra $\mathfrak{k}$ and the subspace $\mathfrak{p}$ satisfy $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$, and also, the subalgebra $\mathfrak{k}$ is the maximal compact subalgebra of $\mathfrak{su}_{p,q}$.

The decomposition of $\mathfrak{su}_{p,q}$ induced by θ is a *Cartan decomposition*. The *compact real form* $\mathfrak{su}_n$ of $\mathfrak{sl}_n(\mathbb{C})$, given by $\mathfrak{su}_n = \{X \in \mathfrak{sl}_n(\mathbb{C}) : X^* = -X\}$, satisfies the properties: $\theta(\mathfrak{su}_n) \subseteq \mathfrak{su}_n$, $\mathfrak{k} = \mathfrak{su}_{p,q} \cap \mathfrak{su}_n$, $\mathfrak{p} = \mathfrak{su}_{p,q} \cap \iota\mathfrak{su}_n$. The involutive automorphism θ is a *Cartan involution*. This Cartan decomposition of the Lie algebra $\mathfrak{su}_{p,q}$ induces a Cartan decomposition of the Lie group $SU_{p,q}$, which is given by $SU_{p,q} = K \exp(\mathfrak{p})$, where $K = \exp(\mathfrak{k}) = S(U_p \otimes U_q) = SU_p \otimes U_1 \otimes SU_q$ is the maximal compact subgroup of $SU_{p,q}$.

The decomposition of $\mathfrak{su}_{p,q}$ into $\mathfrak{k} \oplus \mathfrak{p}$ induces the following similar decomposition on $\mathfrak{sl}_n(\mathbb{C})$. As the complexification of $\mathfrak{su}_{p,q}$,

$$\begin{aligned}\mathfrak{sl}_n(\mathbb{C}) &= \{Z = X + \iota Y : X, Y \in \mathfrak{su}_{p,q}\} \\ &= \{Z = (X_{\mathfrak{k}} + X_{\mathfrak{p}}) + \iota(Y_{\mathfrak{k}} + Y_{\mathfrak{p}}) : X_{\mathfrak{k}}, Y_{\mathfrak{k}} \in \mathfrak{k},\ X_{\mathfrak{p}}, Y_{\mathfrak{p}} \in \mathfrak{p}\} \\ &= \{Z = X_{\mathbf{c}} + Y_{\mathbf{c}} : X_{\mathbf{c}} = X_{\mathfrak{k}} + \iota Y_{\mathfrak{k}} \in \mathfrak{k}_{\mathbf{c}},\ Y_{\mathbf{c}} = X_{\mathfrak{p}} + \iota Y_{\mathfrak{p}} \in \mathfrak{p}_{\mathbf{c}}\}.\end{aligned}$$

That is, $\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{k}_{\mathbf{c}} \oplus \mathfrak{p}_{\mathbf{c}}$, and $\mathfrak{k}_{\mathbf{c}}$ and $\mathfrak{p}_{\mathbf{c}}$ satisfy $[\mathfrak{k}_{\mathbf{c}}, \mathfrak{k}_{\mathbf{c}}] \subseteq \mathfrak{k}_{\mathbf{c}}$, $[\mathfrak{k}_{\mathbf{c}}, \mathfrak{p}_{\mathbf{c}}] \subseteq \mathfrak{p}_{\mathbf{c}}$, $[\mathfrak{p}_{\mathbf{c}}, \mathfrak{p}_{\mathbf{c}}] \subseteq \mathfrak{k}_{\mathbf{c}}$.

The *Iwasawa decomposition* of $\mathfrak{sl}_n(\mathbb{C})$ is given by $\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{su}_n \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{a}$ consists of those members of $\mathfrak{sl}_n(\mathbb{C})$ with real entries on the diagonal and with zeros off the diagonal, and $\mathfrak{n}$ consists of those members of $\mathfrak{sl}_n(\mathbb{C})$ that have zeros on and below the diagonal.

The Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ can also be decomposed as $\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{n}_- \oplus \mathfrak{h}_{\mathrm{d}} \oplus \mathfrak{n}_+$ (triangular decomposition), where $\mathfrak{n}_-(\mathfrak{n}_+)$ is the nilpotent Lie algebra of strictly lower (upper) triangular matrices and $\mathfrak{h}_{\mathrm{d}}$, an abelian Lie algebra of diagonal matrices, is a *Cartan subalgebra* of $\mathfrak{sl}_n(\mathbb{C})$. That is,

$$\mathfrak{h}_{\mathrm{d}} = \{\Lambda = \mathrm{diag}[\lambda_1, \ldots, \lambda_n] : \lambda_i \in \mathbb{C},\ \mathrm{tr}(\Lambda) = 0\}.$$

The *Borel subalgebra* $\mathfrak{b}$, which is a solvable subalgebra of $\mathfrak{sl}_n(\mathbb{C})$, is given by $\mathfrak{b} = \mathfrak{h}_{\mathrm{d}} \oplus \mathfrak{n}_+$, and its derived algebra $[\mathfrak{b}, \mathfrak{b}]$ is $\mathfrak{n}_+$. Any subalgebra of $\mathfrak{sl}_n(\mathbb{C})$ containing a Borel subalgebra is a *parabolic subalgebra* of $\mathfrak{sl}_n(\mathbb{C})$.

3. Root structures and Cartan subgroups

Let E_{ij}, $1 \le i, j \le n$ denote matrix units which are $(n \times n)$ matrices with 1 at the *ij*th entry and zero elsewhere. That is, the *kl* element of E_{ij} is given in terms of the Kronecker delta by $\delta_{ik}\delta_{jl}$. Then $[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}$. A basis of $\mathfrak{sl}_n(\mathbb{C})$ can be given in terms of matrix units E_{ij} by defining the set $\{E_k, E_{ij} : i \ne j,\ i, j = 1, \ldots, n,\ k = 1, \ldots, n-1;\ E_k := E_{kk} - E_{nn},\ E_n = 0\}$. The set $\{E_k\}_{k=1}^{n-1}$ forms a basis of the Cartan subalgebra $\mathfrak{h}_{\mathrm{d}}$ of $\mathfrak{sl}_n(\mathbb{C})$. It will be useful in what follows to consider the following alternative basis for $\mathfrak{sl}_n(\mathbb{C})$. Let, for $i, j = 1, \ldots, n$,

$$X_i = E_i - \frac{1}{n}\sum_{k=1}^{n-1} E_k, \quad X_{ij} := E_{ij},\ i \ne j. \tag{3.1}$$

Note that $X_1 + \cdots + X_n = 0$. Clearly, the set of all X_{ij}, $i \neq j$ with any $(n-1)$ of the X_k forms a basis of $\mathfrak{s}l_n(\mathbb{C})$. These basis elements satisfy the commutation relations $[X_{ij}, X_{kl}] = \delta_{jk}X_{il} - \delta_{li}X_{kj}$, $X_{mm} = X_m$.

Let us now consider the root structure of $\mathfrak{s}l_n(\mathbb{C})$ relative to the Cartan subalgebra $\mathfrak{h}_\mathrm{d}$. Denote by $\mathfrak{h}_\mathrm{d}^*$ the dual space consisting of all $\mathbb{C}$-valued linear forms α_u on $\mathfrak{h}_\mathrm{d}$, such that

$$\alpha_u(\Lambda) = \sum_{m=1}^{n} u_m\lambda_m, \quad \Lambda \in \mathfrak{h}_\mathrm{d}, \tag{3.2}$$

where $u = (u_1, \ldots, u_n) \in \mathbb{C}^n$. One has that $\alpha_{-u} = -\alpha_u$. Let $\mathfrak{g}_{\alpha_u}$ denote, for every $\alpha_u \in \mathfrak{h}_\mathrm{d}^*$, the linear subspace of $\mathfrak{s}l_n(\mathbb{C})$ defined by $\mathfrak{g}_{\alpha_u} = \{Y \in \mathfrak{s}l_n(\mathbb{C}) : \mathrm{ad}(\Lambda)Y = \alpha_u(\Lambda)Y\,\forall\Lambda \in \mathfrak{h}_\mathrm{d}\}$. Note that $\mathfrak{g}_{\alpha_u} = \mathfrak{h}_\mathrm{d}$ when $\alpha_u \equiv 0$. Now define the subset R of $\mathfrak{h}_\mathrm{d}^*$ by

$$\begin{aligned} R &= \{\alpha_u \in \mathfrak{h}_\mathrm{d}^* : \alpha_u \not\equiv 0,\ \mathfrak{g}_{\alpha_u} \neq \{0\}\} \\ &= \{\alpha_u \in \mathfrak{h}_\mathrm{d}^* : u = (0, \ldots, 0, 1_i, 0, \ldots, 0, -1_j, 0, \ldots, 0),\ 1 \le i \neq j \le n\}, \quad \text{or} \\ &\quad\ \{\alpha_u \in \mathfrak{h}_\mathrm{d}^* : u = (0, \ldots, 0, -1_i, 0, \ldots, 0, 1_j, 0, \ldots, 0),\ 1 \le i \neq j \le n\}. \end{aligned} \tag{3.3}$$

The set R is finite and its elements are the non-zero roots of $\mathfrak{s}l_n(\mathbb{C})$ relative to $\mathfrak{h}_\mathrm{d}$. Thus, R is called the root system of the pair $(\mathfrak{s}l_n(\mathbb{C}), \mathfrak{h}_\mathrm{d})$. For every root α_u, $\mathfrak{g}_{\alpha_u}$ is of dimension 1. One has then the *root space decomposition* of $\mathfrak{s}l_n(\mathbb{C})$ given by

$$\mathfrak{s}l_n(\mathbb{C}) = \mathfrak{h}_\mathrm{d} \oplus \sum_{\alpha_u \in R}^{\oplus} \mathfrak{g}_{\alpha_u} = \mathfrak{h}_\mathrm{d} \oplus \sum_{\substack{1 \le i, j \le n \\ i \neq j}} \{\mathbb{C}E_{ij}\}. \tag{3.4}$$

Allowing α_{ij} to denote the form α_u where $u = (0, \ldots, 0, 1_i, 0, \ldots, 0, -1_j, 0, \ldots, 0)$, one defines R^+ to be the collection $\{\alpha_{ij} : 1 \le i < j \le n\}$. R^+ is the set of positive roots of $\mathfrak{s}l_n(\mathbb{C})$ relative to $\mathfrak{h}_\mathrm{d}$, each of which can be written as the sum of fundamental roots $\alpha_{i\,i+1}$, $1 \le i \le n-1$. Clearly, $R = R^+ \cup (-R^+)$, where $-R^+$ is the collection of all *negative roots*. That is $-R^+ = \{-\alpha_{ij} = \alpha_{ji} : 1 \le i < j \le n\}$. Thus each non-trivial linear subspace $\mathfrak{g}_{\alpha_u}$ is given by $\mathfrak{g}_{\alpha_{ij}} = \{\mathbb{C}E_{ij}\}_{i \neq j}$. Note also that $[\Lambda, E_{ij}] = (\lambda_i - \lambda_j)E_{ij} = \alpha_{ij}(\Lambda)E_{ij}\,\forall\Lambda \in \mathfrak{h}_\mathrm{d}$.

A root α_{ij} is called compact if $i, j \le p$ or $i, j > p$, otherwise it is said to be non-compact. The set of all compact roots is denoted by R_c. Each $\mathfrak{g}_{\alpha_{ij}}$ for which $\alpha_{ij} \in R_\mathrm{c}$ is a subspace of $\mathfrak{k}_\mathbf{c}$, and each $\mathfrak{g}_{\alpha_{ij}}$ for which $\alpha_{ij} \in R - R_\mathrm{c}$ is a subspace of $\mathfrak{p}_\mathbf{c}$.

As defined in [16], two roots $\alpha_{kl}, \beta_{st} \in R$, $0 \le k \neq l \le n$, $0 \le s \neq t \le n$ are said to be *strongly orthogonal* if $\alpha_{kl} \pm \beta_{st} \notin R$. Let Γ_j, $0 \le j \le p$ be a subset of $R^+ \cap (R - R_\mathrm{c})$ consisting only of j strongly orthogonal positive non-compact roots ($\Gamma_0 = \emptyset$). Such a subset is called a system of strongly orthogonal positive non-compact roots. Two such systems Γ_{j_1} and Γ_{j_2} are said to be *equivalent* if $\sum_{\alpha_{kl} \in \Gamma_{j_1}} \mathbb{R}\Lambda_{\alpha_{kl}} = \sum_{\alpha_{kl} \in \Gamma_{j_2}} \mathbb{R}\Lambda_{\alpha_{kl}}$, where $\Lambda_{\alpha_{kl}}$ is the unique element in $\mathfrak{h}_\mathrm{d}$ such that the Killing form $B(\Lambda, \Lambda_{\alpha_{kl}}) = 2n\,\mathrm{tr}(\Lambda\Lambda_{\alpha_{kl}}) = \alpha_{kl}(\Lambda)$ for all $\Lambda \in \mathfrak{h}_\mathrm{d}$. That is, $\Lambda_{\alpha_{kl}} = \mathrm{diag}[0, \ldots, 0, 1_k/2n, 0, \ldots, 0, -1_l/2n, 0, \ldots, 0]$. Two such systems are said to be *conjugate* if there exists an element w in the Weyl group of the pair $(\mathfrak{s}l_n(\mathbb{C}), \mathfrak{h}_\mathrm{d})$ such that $w\Gamma_{j_1}$ is equivalent to Γ_{j_2}. Such an element w exists whenever $j_1 = j_2$. This conjugacy induces an equivalence relation on the set of all Γ_j, $0 \le j \le p$. One can see that, for $p \le q$, there exist $(p+1)$ such conjugacy classes, denoted by $\bar{\Gamma}_j,\ 0 \le j \le p$. For example, an elements Γ_j of $\bar{\Gamma}_j$ can be given by, for $1 \le j \le p$, $\Gamma_j = \{\alpha_{kl} : k = p+1-m,\ l = p+m,\ 1 \le m \le j\}$, and $\Gamma_0 = \emptyset$.

We now define subspaces in terms of the elements Γ_j of $\bar{\Gamma}_j$. That is, for $1 \le j \le p$, let $\mathfrak{h}_j^{\mathrm{v}} = \sum_{\alpha_{kl}\in\Gamma_j} \mathbb{R}(E_{\alpha_{kl}} + E_{-\alpha_{kl}})$, where $E_{\alpha_{kl}}$, denoting E_{kl}, be a subspace of $\mathfrak{p}$ in the Cartan decomposition of $\mathfrak{su}_{p,q}$. Clearly, the subspaces $\mathfrak{h}_j^{\mathrm{v}}$ determined by different elements $\Gamma_j \in \bar{\Gamma}_j$ form a conjugacy class under the actions of the Weyl group $W(\mathfrak{sl}_n(\mathbb{C}), \mathfrak{h}_{\mathrm{d}})$. That is, there exists $w \in W(\mathfrak{sl}_n(\mathbb{C}), \mathfrak{h}_{\mathrm{d}})$ such that, for $1 \le j \le p$,

$$\mathfrak{h}_j^{\mathrm{v}} = \sum_{\alpha_{kl}\in\Gamma_{(j)}} \mathbb{R}(E_{\alpha_{kl}} + E_{-\alpha_{kl}}) \cong \sum_{\alpha_{kl}\in w\Gamma_{(j)}} \mathbb{R}(E_{\alpha_{kl}} + E_{-\alpha_{kl}}) \cong w(\mathfrak{h}_j^{\mathrm{v}}) \cong \mathfrak{h}_j^{\mathrm{v}'}.$$

Choosing the elements Γ_j as above, the subspaces $\mathfrak{h}_j^{\mathrm{v}},\ 1 \le j \le p$ then consist of $n \times n$ matrices of the form

$$\begin{pmatrix} & & & & & t_j \\ & & & & \iddots & \\ & & & t_1 & & \\ & & t_1 & & & \\ & \iddots & & & & \\ t_j & & & & & \end{pmatrix},$$

where $t_1, \dots, t_j \in \mathbb{R}$, and, one defines $\mathfrak{h}_0^{\mathrm{v}}$ as a singleton containing the zero matrix. Note also that the eigenvalues of $\mathrm{ad}(X)$ for any $X \in \mathfrak{h}_j^{\mathrm{v}}$ are all real.

The abelian subalgebras $\mathfrak{h}_j^{\mathrm{t}},\ 0 \le j \le p$ are defined by

$$\mathfrak{h}_j^{\mathrm{t}} = \{\mathrm{diag}[\iota\phi_1, \dots, \iota\phi_{p-j}, \iota\theta_j, \dots, \iota\theta_1, \iota\theta_1, \dots, \iota\theta_j, \iota\psi_{q-j}, \dots, \iota\psi_1] : \phi_k, \theta_k, \psi_k$$

$$\in \mathbb{R}, \phi_1 + \cdots + \phi_{p-j} + 2(\theta_j + \cdots + \theta_1) + \psi_{q-j} + \cdots + \psi_1 = 0\}.$$

Note that the eigenvalues of $\mathrm{ad}(X)$ for any $X \in \mathfrak{h}_j^{\mathrm{t}}$ are all imaginary. Each $\mathfrak{h}_j = \mathfrak{h}_j^{\mathrm{t}} \oplus \mathfrak{h}_j^{\mathrm{v}},\ 0 \le j \le p$ is a maximal abelian subalgebra of $\mathfrak{su}_{p,q}$. The subalgebras $\mathfrak{h}_j^{\mathrm{t}}$ and the subspaces $\mathfrak{h}_j^{\mathrm{v}}$ are often called the *toroidal* and the *vector* parts of $\mathfrak{h}_j$, respectively. The real dimension of $\mathfrak{h}_j$ is $(n-1)$, which is equal to the rank of $\mathfrak{su}_{p,q}$. Also, $\dim(\mathfrak{h}_j \cap \mathfrak{k}) = (n-j-1)$ and $\dim(\mathfrak{h}_j \cap \mathfrak{p}) = j$. The algebra $\mathfrak{h}_j$ is maximal abelian in the sense that there exists no other abelian subalgebra of $\mathfrak{su}_{p,q}$ with real dimension greater than the real dimension of $\mathfrak{h}_j$. Each $\mathfrak{h}_j$ satisfies the property that, for every $X \in \mathfrak{h}_j$, $\mathrm{ad}(X)$ is a semisimple (diagonalizable) endomorphism of $\mathfrak{su}_{p,q}$. That is, the subalgebras $\mathfrak{h}_j$ are *Cartan subalgebras* of $\mathfrak{su}_{p,q}$. We say that two Cartan subalgebras $\mathfrak{h}'_j, \mathfrak{h}''_j$ are conjugate if their respective vector parts lie in the same conjugacy class. Equivalently, one may say that $\mathfrak{h}'_j$ and $\mathfrak{h}''_j$ are conjugate if $\mathfrak{h}''_j = \mathrm{Ad}(g)\mathfrak{h}'_j$ for some $g \in SU_{p,q}$. Thus there exist $(p+1)$ conjugacy classes of Cartan subalgebras for $\mathfrak{su}_{p,q}$. This is an instance of the Kostant–Sugiura Theorem [25,29], which follows the following theorem.

Theorem 3.1. *There is a one-to-one correspondence between the conjugacy classes of Cartan subalgebras in a real semisimple Lie algebra* $\mathfrak{g}$ *and the conjugacy classes of strongly orthogonal systems of positive non-compact roots of the pair* $(\mathfrak{g}_{\mathbf{c}}, \mathfrak{h}_{\mathbf{c}})$, *where* $\mathfrak{g}_{\mathbf{c}}$ *is the complexification of* $\mathfrak{g}$ *and* $\mathfrak{h}_{\mathbf{c}}$ *is a Cartan subalgebra of* $\mathfrak{g}_{\mathbf{c}}$.

Because any two Cartan subalgebras of $\mathfrak{sl}_n(\mathbb{C})$ are conjugate under an automorphism of $\mathfrak{sl}_n(\mathbb{C})$, the complexifications of the Cartan subalgebras of $\mathfrak{su}_{p,q}$, regardless of their conjugacy class, fall into the single conjugacy class of Cartan subalgebras in $\mathfrak{sl}_n(\mathbb{C})$ (which contains $\mathfrak{h}_{\mathrm{d}}$). Also, where $\mathfrak{h}_j^{\mathbf{c}}$ is any Cartan subalgebra of $\mathfrak{sl}_n(\mathbb{C})$, the construction of roots $\alpha_{u_{(j)}}$ of $(\mathfrak{sl}_n(\mathbb{C}), \mathfrak{h}_j^{\mathbf{c}})$ exactly parallels the construction given earlier of the roots α_u of $(\mathfrak{sl}_n(\mathbb{C}), \mathfrak{h}_{\mathrm{d}})$. In particular, Eq. (3.2) would become

$$\alpha_{u_{(j)}}(X_{(j)}) = \sum_{m=1}^{n} u_{(j)m}\lambda_{(j)m}, \qquad X_{(j)} \in \mathfrak{h}_j^{\mathbf{c}}, \tag{3.5}$$

where $u_{(j)} = (u_{(j)1}, \dots, u_{(j)n}) \in \mathbb{C}^n$ and $\lambda_{(j)m}$ are the eigenvalues of $X_{(j)}$ under some suitable ordering.

The subgroup of $SU_{p,q}$ generated by $\mathfrak{h}_j^{\mathrm{t}}$ is a *toroidal group* denoted by $H_j^{\mathrm{t}} = \exp(\mathfrak{h}_j^{\mathrm{t}})$, and the subgroup generated by $\mathfrak{h}_j^{\mathrm{v}}$ is a *vector group* denoted by $H_j^{\mathrm{v}} = \exp(\mathfrak{h}_j^{\mathrm{v}})$. In terms of the toroidal group H_j^{t} and the vector group H_j^{v} one has $H_j = H_j^{\mathrm{v}} H_j^{\mathrm{t}}$. Thus, for $p \le q$, the group $SU_{p,q}$ has $(p+1)$ conjugacy classes of Cartan subgroups, denoted by $\bar{H}_j$, $0 \le j \le p$. That is, the elements of a class $\bar{H}_j$ are conjugate under an inner automorphism of $SU_{p,q}$. One may realize a representative element H_j of the class $\bar{H}_j$ by $H_j = \{h_{(j)}\}$ such that

$$h_{(j)} = \begin{pmatrix} I_{p-j} &&&&&&& \\ & \cosh t_j &&&&& \sinh t_j & \\ && \ddots &&& \iddots && \\ &&& \cosh t_1 & \sinh t_1 &&& \\ &&& \sinh t_1 & \cosh t_1 &&& \\ && \iddots &&& \ddots && \\ & \sinh t_j &&&&& \cosh t_j & \\ &&&&&&& I_{q-j} \end{pmatrix}$$

$$\times \mathrm{diag}[\mathrm{e}^{\iota\phi_1}, \dots, \mathrm{e}^{\iota\phi_{p-j}}; \mathrm{e}^{\iota\theta_j}, \dots, \mathrm{e}^{\iota\theta_1}; \mathrm{e}^{\iota\theta_1}, \dots, \mathrm{e}^{\iota\theta_j}; \mathrm{e}^{\iota\psi_{q-j}}, \dots, \mathrm{e}^{\iota\psi_1}], \tag{3.6}$$

where $\sum_{n=1}^{p-j}\phi_n + 2\sum_{n=1}^{j}\theta_n + \sum_{n=1}^{q-j}\psi_n = 0$, and $t_j, \phi_n, \theta_n, \psi_n \in \mathbb{R}$. Thus after multiplication, one has

$$h_{(j)} = \begin{pmatrix} e^{\iota\phi_1} &&&&&&&&&& \\ & \ddots &&&&&&&&& \\ && e^{\iota\phi_{p-j}} &&&&&&&& \\ &&& \omega_j^+ &&&& \omega_j^- &&& \\ &&&& \ddots && \iddots &&&& \\ &&&&& \omega_1^+ \; \omega_1^- &&&&& \\ &&&&& \omega_1^- \; \omega_1^+ &&&&& \\ &&&& \iddots && \ddots &&&& \\ &&& \omega_j^- &&&& \omega_j^+ &&& \\ &&&&&&&& e^{\iota\psi_{q-j}} && \\ &&&&&&&&& \ddots & \\ &&&&&&&&&& e^{\iota\psi_1} \end{pmatrix}, \tag{3.7}$$

where $\omega_k^{\pm} = \frac{1}{2}e^{\iota\theta_k}(e^{t_k} \pm e^{-t_k})$. The matrices $h_{(j)}$ are normal matrices since $h^*_{(j)}h_{(j)} = h_{(j)}h^*_{(j)}$. The eigenvalues of $h_{(j)} \in H_j$, $0 \le j \le p$ are given as the elements of the n-tuple

$$\begin{aligned}(h_{(j)1}, \dots, h_{(j)n}) \\ = (e^{\iota\phi_1}, \dots, e^{\iota\phi_{p-j}}; e^{z_j}, \cdots, e^{z_1}; e^{-\bar{z}_1}, \dots, e^{-\bar{z}_j}; e^{\iota\psi_{q-j}}, \dots, e^{\iota\psi_1}),\end{aligned} \tag{3.8}$$

where $z_m = \iota\theta_m + t_m$, $1 \le m \le j$. Clearly, the Cartan subgroup H_j is homeomorphic to $\mathbb{T}^{n-j} \times \mathbb{R}^j$, where $\mathbb{T}^m$ is an m-dimensional torus.

4. Characters and density functions

Let H be a Cartan subgroup of a semisimple real Lie group G and let $G_{\mathbf{c}}$, $H_{\mathbf{c}}$ denote, respectively, the complexification of G and H. Let $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{g}_{\mathbf{c}}$, and $\mathfrak{h}_{\mathbf{c}}$ denote the corresponding Lie algebras, and let $\mathfrak{u}$ be a compact real form of $\mathfrak{g}_{\mathbf{c}}$. Also, let $H_\alpha \in \iota(\mathfrak{h}_{\mathbf{c}} \cap \mathfrak{u})$ such that $B(X, H_\alpha) = \alpha(X)$ for all $X \in \mathfrak{h}_{\mathbf{c}}$, where α is a root of $(\mathfrak{g}_{\mathbf{c}}, \mathfrak{h}_{\mathbf{c}})$. Then, with this notation, one has the following well-known theorem by Weyl [7].

Theorem 4.1. *Where $G_{\mathbf{c}}$ is simply connected and λ is any linear function on $\mathfrak{h}_{\mathbf{c}}$, there exists at most one character (complex analytic homomorphism) ξ_λ of $H_{\mathbf{c}}$ such that $\xi_\lambda(\exp(X)) = e^{\lambda(X)}$, $X \in \mathfrak{h}_{\mathbf{c}}$ if and only if $2\lambda(H_\alpha)/\alpha(H_\alpha)$ is an integer for every root α of $(\mathfrak{g}_{\mathbf{c}}, \mathfrak{h}_{\mathbf{c}})$.*

This condition on λ is equivalent to the condition that $\lambda(X) \in 2\pi\iota\mathbb{Z}$, whenever X satisfies $e^X = 1$. Clearly, where λ is a root of $(\mathfrak{g}_{\mathbf{c}}, \mathfrak{h}_{\mathbf{c}})$, ξ_λ is defined. Also, where R^+ denotes the set of all positive roots of $(\mathfrak{g}_{\mathbf{c}}, \mathfrak{h}_{\mathbf{c}})$, and $\delta = \frac{1}{2}\sum_{\alpha \in R^+}\alpha$, the character ξ_δ exists. According to Harish-Chandra [7], G is said to be *acceptable* if ξ_δ can be defined on $H_{\mathbf{c}}$. Thus in the above case, where $G_{\mathbf{c}}$ is simply connected, G is acceptable. Also, a linear functional λ of $\mathfrak{h}_{\mathbf{c}}$ that is real-valued on $\mathfrak{h}$ is said to be *dominant* if $2\lambda(H_\alpha)/\alpha(H_\alpha) \geq 0\,\forall\alpha \in R^+$.

Let us now consider $\mathfrak{sl}_n(\mathbb{C})$. Let $\mathfrak{h}_j^{\mathbf{c}}$ denote the complexification of $\mathfrak{h}_j$. Because $\mathfrak{sl}_n(\mathbb{C})$ is simply connected, $SU_{p,q}$ is acceptable. Hence, one may define a root of $(SU_{p,q}, H_j)$ as the character $\xi_{(j)\alpha_{u_{(j)}}} = \hat{\xi}_{(j)\alpha_{u_{(j)}}} \circ \iota$ of H_j into $\mathbb{C} - \{0\}$ corresponding to the root $\alpha_{u_{(j)}}$ of $(\mathfrak{sl}_n(\mathbb{C}), \mathfrak{h}_j^{\mathbf{c}})$. Here ι is the inclusion mapping from H_j into its complexification $H_j^{\mathbf{c}}$ (which is given by $\mathrm{e}^{\mathfrak{h}_j^{\mathbf{c}}}$), and $\hat{\xi}_{(j)\alpha_{u_{(j)}}}$ is the character of $H_j^{\mathbf{c}}$ corresponding to $\alpha_{u_{(j)}}$. That is, from Theorem 4.1,

$$\xi_{(j)\alpha_{u_{(j)}}}(h_{(j)}) = \mathrm{e}^{\alpha_{u_{(j)}}(\ln(h^{\mathbf{c}}_{(j)}))}, \quad h^{\mathbf{c}}_{(j)} = \iota(h_{(j)}). \tag{4.1}$$

We consider the following four cases:

1. The set of all *positive roots* of $(SU_{p,q}, H_j)$ is given by
$$\begin{aligned} R^+_{(j)} &= \{\xi_{(j)kl} : 1 \leq k < l \leq n,\ kl\,\text{denotes}\,\alpha_{u_{(j)}},\ u_{(j)} \\ &\quad = (0,\dots,0,1_k,0,\dots,0,-1_l,0,\dots,0)\}, \\ |R^+_{(j)}| &= \tfrac{1}{2}n(n-1),\ \xi_{(j)kl}(h_{(j)}) = h_{(j)k}h^{-1}_{(j)l}, \end{aligned} \tag{4.2}$$
where $h_{(j)k}$ and $h_{(j)l}$ are eigenvalues of H_j under some suitable ordering.
2. The subset of all *compact positive roots* of $(SU_{p,q}, H_j)$, which are necessarily imaginary (i.e., the corresponding root of $(\mathfrak{sl}_n(\mathbb{C}), \mathfrak{h}_j^{\mathbf{c}})$ has an imaginary image), is given by
$$\begin{aligned} R^+_{(j)\mathrm{c}} &= \{\xi_{(j)kl}, \xi_{(j)mn} : 1 \leq k < l \leq p-j,\ 1 \leq n < m \leq q-j\}, \\ |R^+_{(j)\mathrm{c}}| &= \tfrac{1}{2}n(n-1) - pq + j^2 - j(n-1),\ \xi_{(j)kl}(h_{(j)}) \\ &\quad = h_{(j)k}h^{-1}_{(j)l} = \mathrm{e}^{\mathrm{i}(\phi_{(j)k}-\phi_{(j)l})},\ \xi_{(j)mn}(h_{(j)}) = h_{(j)m}h^{-1}_{(j)n} = \mathrm{e}^{\mathrm{i}(\psi_{(j)m}-\psi_{(j)n})}. \end{aligned} \tag{4.3}$$
3. The subset of all *real positive roots* of $(SU_{p,q}, H_j)$, which are necessarily singular (non-compact), is given by
$$\begin{aligned} R^+_{(j)\mathrm{R}} &= \{\xi_{(j)kl} : k = p-j+s,\ l = p+j-s+1,\ 1 \leq s \leq j\}, \\ |R^+_{(j)\mathrm{R}}| &= j,\ \xi_{(j)kl}(h_{(j)}) = h_{(j)k}h^{-1}_{(j)l} = \mathrm{e}^{z_{(j)m}+\bar{z}_{(j)m}} = \mathrm{e}^{2t_{(j)m}},\ m = \tfrac{1}{2}(l-k+1). \end{aligned} \tag{4.4}$$
4. The subset of all *singular imaginary positive roots* of $(SU_{p,q}, H_j)$ is given by
$$\begin{aligned} R^+_{(j)\mathrm{SI}} &= \{\xi_{(j)kl} : 1 \leq k \leq p-j,\ p+j+1 \leq l \leq n\}, \\ |R^+_{(j)\mathrm{SI}}| &= pq - j(n-j),\ \xi_{(j)kl}(h_{(j)}) = h_{(j)k}h^{-1}_{(j)l} = \mathrm{e}^{\mathrm{i}(\phi_{(j)k}-\psi_{(j)l})}. \end{aligned} \tag{4.5}$$

All the remaining $2j(n-j-1)$ positive roots of $(SU_{p,q}, H_j)$ are *singular complex* and they are given by the elements of the complement of $(R^+_{(j)\mathrm{c}} \cup R^+_{(j)\mathrm{R}} \cup R^+_{(j)\mathrm{SI}})$ in $R^+_{(j)}$.

The Weyl group of $(SU_{p,q}, H_j)$ is defined by $W(SU_{p,q}, H_j) = \{\sigma | H_j : \sigma \in \mathrm{Inn}(SU_{p,q}), \sigma(H_j) = H_j\}$, which is isomorphic to the group generated by the elements in the set $S_{p-j} \cup S_{q-j} \cup S_j \cup \mathbf{P}(j)$, where

1. S_{p-j} is the symmetric group consisting of all permutations of $e^{i\phi_k}$, $1 \le k \le p-j$.
2. S_{q-j} is the symmetric group consisting of all permutations of $e^{i\psi_k}$, $1 \le k \le q-j$.
3. S_j is the symmetric group consisting of all permutations of the j pairs $(e^{z_k}, e^{-\bar{z}_k})$, $1 \le k \le j$. That is, the permutation of the pair $(e^{z_k}, e^{-\bar{z}_k})$ and $(e^{z_l}, e^{-\bar{z}_l})$ corresponds to $e^{z_k} \leftrightarrow e^{z_l}$ and $e^{-\bar{z}_k} \leftrightarrow e^{-\bar{z}_l}$.
4. $\mathbf{P}(j)$ is the power set of the set consisting of the j permutations of e^{z_k} and $e^{-\bar{z}_k}$, $1 \le k \le j$ (i.e., the change of sign on t_k in $z_k = t_k + i\theta_k$ and $-\bar{z}_k = -t_k + i\theta_k$).

Thus, we have $|W(SU_{p,q}), H_j)| = (p-j)!(q-j)!j!2^j$. Now, letting $\alpha_{u_{(j)}} = \delta$, where, for $\mathfrak{sl}_n(\mathbb{C})$,

$$\delta = \frac{1}{2}\sum_{\substack{i,j=1\\ i<j}}^{n} \alpha_{ij} = \frac{1}{2}\sum_{k=1}^{n}(n-2k+1)\mathbf{e}_k, \tag{4.6}$$

we have from Eqs. (3.2), (3.8), (4.1) and (4.2) that

$$\begin{aligned}\xi_{(j)\delta}(h_{(j)}) &= \prod_{R^+_{(j)}} (\xi_{(j)kl}(h_{(j)}))^{1/2} = \prod_{k=1}^{n} (h_{(j)k}^{(n-2k+1)})^{1/2} \\ &= \prod_{k=1}^{p-j} e^{(i/2)\phi_{(j)k}(n-2k+1)} \prod_{k=p-j+1}^{p} e^{(1/2)z_{(j)p+1-k}(n-2k+1)} \\ &\quad\times \prod_{k=p+1}^{p+j} e^{(-1/2)\bar{z}_{(j)k-p}(n-2k+1)} \prod_{k=p+j+1}^{n} e^{(i/2)\psi_{(j)n+1-k}(n-2k+1)},\end{aligned} \tag{4.7}$$

where by the unimodularity condition $h_{(j)n} = (h_{(j)1}h_{(j)2}\cdots h_{(j)n-1})^{-1}$, $\xi_{(j)\delta}$ becomes single-valued. Using this condition explicitly, one obtains

$$\begin{aligned}\xi_{(j)\delta}(h_{(j)}) &= \prod_{k=1}^{n-1} h_{(j)k}^{n-k} = \prod_{k=1}^{p-j} e^{i\phi_{(j)k}(n-k)} \prod_{k=p-j+1}^{p} e^{z_{(j)p+1-k}(n-k)} \prod_{k=p+1}^{p+j} e^{-\bar{z}_{(j)k-p}(n-k)} \\ &\quad\times \prod_{k=p+j+1}^{n-1} e^{i\psi_{(j)n+1-k}(n-k)}.\end{aligned}$$

Now, following Harish-Chandra [7], one may define the *density functions* Δ for $SU_{p,q}$ as

$$\begin{aligned}\Delta_{(j)}(h_{(j)}) &= \xi_{(j)\delta}(h_{(j)})\prod_{R^+_{(j)}}(1-\xi_{(j)kl}(h^{-1}_{(j)})) = \prod_{1\le k<l\le n}(h_{(j)k}-h_{(j)l}) \\ &= (-1)^{(1/2)n(n-1)}\det\begin{pmatrix} 1 & \cdots & 1 \\ h_{(j)1} & \cdots & h_{(j)n} \\ h^2_{(j)1} & \cdots & h^2_{(j)n} \\ \vdots & \cdots & \vdots \\ h^{n-1}_{(j)1} & \cdots & h^{n-1}_{(j)n}\end{pmatrix} \\ &= \sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)h^{\sigma(n)-1}_{(j)1}h^{\sigma(n-1)-1}_{(j)2}\cdots h^{\sigma(1)-1}_{(j)n}.\end{aligned} \tag{4.8}$$

The value $\Delta_{(j)}(h_j)$ is called the *Weyl denominator* [30,32].

Theorem 4.2. *The density functions Δ have the following special values*:

$$\begin{aligned}\Delta_{(j)\mathrm{R}}(h_{(j)}) &= \prod_{R^+_{(j)\mathrm{R}}}(1-\xi_{(j)kl}(h^{-1}_{(j)})) = \prod_{k=p-j+1}^{p}(1-h^{-1}_{(j)k}h_{(j)2p+1-k}), \\ \Delta_{(j)\mathrm{SI}}(h_{(j)}) &= \prod_{R^+_{(j)\mathrm{SI}}}(1-\xi_{(j)kl}(h^{-1}_{(j)})) = \prod_{k=1}^{p-j}\prod_{l=p+j+1}^{n}(1-h^{-1}_{(j)k}h_{(j)l}), \\ \Delta_{(j)\mathrm{I}}(h_{(j)}) &= \xi_{(j)\delta}({}^{\mathrm{t}}h_{(j)})\prod_{R^+_{(j)\mathrm{I}}}(1-\xi_{(j)kl}(h^{-1}_{(j)})) \\ &= \prod_{k=p-j+1}^{p+j}({}^{\mathrm{t}}h_{(j)k})^{n-k}\prod_{k=1}^{p-j}h^{2j}_{(j)k}\prod_{1\le k<l\le p-j}(h_{(j)k}-h_{(j)l}) \\ &\quad\times\prod_{\substack{1\le k\le p-j\\ p+j+1\le l\le n}}(h_{(j)k}-h_{(j)l})\prod_{p+j+1\le k<l\le n}(h_{(j)k}-h_{(j)l}), \\ \Delta_{(j)\mathrm{R}\cup\mathrm{C}}(h_{(j)}) &= \xi_{(j)\delta}({}^{\mathrm{v}}h_{(j)})\prod_{R^+_{(j)\mathrm{R}}\cup R^+_{(j)\mathrm{C}}}(1-\xi_{(j)kl}(h^{-1}_{(j)})) \\ &= \prod_{k=p-j+1}^{p+j}({}^{\mathrm{v}}h_{(j)k}h^{-1}_{(j)k})^{n-k}\prod_{k=1}^{p-j}h^{-2j}_{(j)k}\prod_{\substack{1\le k\le p-j\\ p-j+1\le l\le p+j}}(h_{(j)k}-h_{(j)l}) \\ &\quad\times\prod_{\substack{p-j+1\le k\le p+j\\ p-j+2\le l\le n}}(h_{(j)k}-h_{(j)l}), \qquad \Delta_{(j)} = \Delta_{(j)\mathrm{I}}\Delta_{(j)\mathrm{R}\cup\mathrm{C}},\end{aligned}$$

where the indices t *and* v *denote toroidal and vector parts, respectively*; $R^+_{(j)\mathrm{I}}$ *denotes the set of all imaginary positive roots which, when restricted to* H^{v}_j, *vanish identically; and* $R^+_{(j)\mathrm{R}\cup\mathrm{C}}$ *denotes the set of all real and complex positive roots which do not vanish on* H^{v}_j.

Proof. Follows by direct computations from Eqs. (4.2)–(4.5). □

Theorem 4.3. *The density functions Δ satisfy the following conjugation properties*:

1. $\overline{\Delta_{H_j}(h_{(j)})} = (-1)^{|R^+_{(j)}|+|R^+_{(j)\mathrm{R}}|}\Delta_{H_j}(h_{(j)}) = (-1)^{(1/2)n(n-1)+j}\Delta_{H_j}(h_{(j)})$.
2. $\overline{\Delta_{(j)\mathrm{R}}(h_{(j)})} = \Delta_{(j)\mathrm{R}}(h_{(j)})$.
3. $\overline{\Delta_{(j)\mathrm{SI}}(h_{(j)})} = \Delta_{(j)\mathrm{SI}}(h^{-1}_{(j)})$.
4. $\overline{\Delta_{(j)\mathrm{I}}(h_{(j)})} = (-1)^{(1/2)n(n-1)+j}\left(\prod_{k=p-j+1}^{p+j}({}^{\mathrm{t}}h_{(j)k})^{-2(n-k)}\prod_{k=1}^{p-j}h^{-4j}_{(j)k}/(h_{(j)1}\cdots h_{(j)p-j}h_{(j)p+j+1}\cdots h_{(j)n})^{n-2j-1}\right)\Delta_{(j)\mathrm{I}}(h_{(j)})$.
5. $\overline{\Delta_{(j)\mathrm{R}\cup\mathrm{C}}(h_{(j)})} = \left(\prod_{k=p-j+1}^{p+j}({}^{\mathrm{t}}h_{(j)k})^{2(n-k)}\prod_{k=1}^{p-j}h^{4j}_{(j)k}/(h_{(j)1}\cdots h_{(j)p-j}h_{(j)p+j+1}\cdots h_{(j)n})^{2j}(h_{(j)p-j+1}\cdots h_{(j)p+j})^{n-1}\right)\Delta_{(j)\mathrm{R}\cup\mathrm{C}}(h_{(j)})$.

Proof. Follows by direct computations. □

Let $H'_{(j)}$, $H'_{(j)\mathrm{R}}$, $H'_{(j)\mathrm{SI}}$, $H'_{(j)\mathrm{I}}$, and $H'_{(j)\mathrm{R}\cup\mathrm{C}}$ be the subsets of H_j defined by all $h_{(j)} \in H_j$ such that $\Delta_{(j)}(h_{(j)}) \neq 0$, $\Delta_{(j)\mathrm{R}}(h_{(j)}) \neq 0$, $\Delta_{(j)\mathrm{SI}}(h_{(j)}) \neq 0$, $\Delta_{(j)\mathrm{I}}(h_{(j)}) \neq 0$, and $\Delta_{(j)\mathrm{R}\cup\mathrm{C}}(h_{(j)}) \neq 0$, respectively. Let $SU'_{p,q(j)} = \{gh'_{(j)}g^{-1} : g \in SU_{p,q},\ h'_{(j)} \in H'_{(j)}\}$, then one has $SU'_{p,q} = \cup_{j=0}^{p} SU'_{p,q(j)}$. The elements of $SU'_{p,q}$ are said to be *regular elements* in $SU_{p,q}$. Clearly $H'_{(j)} = H_j \cap SU'_{p,q}$. One may also define the regular elements of $SU_{p,q}$ as follows. Consider the *characteristic polynomials* of $\mathrm{Ad}(g)$, where $g \in SU_{p,q}$, $\det((t+1)I - \mathrm{Ad}(g)) = \sum_{0\le k\le n^2-1} D_k(g)t^k$, where t is an indeterminate, $(n^2-1) = \dim(SU_{p,q})$, and D_k are analytic functions on $SU_{p,q}$ with $D_{n^2-1} = 1$. An element $g \in SU_{p,q}$ is said to be regular if $D_l(g) \neq 0$ for $l = \mathrm{rank}\, SU_{p,q} = \mathrm{rank}(\mathfrak{su}_{p,q}) = (n-1)$ and *singular* if $D_l(g) = 0$. For the case when $g = h'_{(j)} \in H'_{(j)}$ one can show by direct computation $D_{n-1}(h'_{(j)}) = (-1)^{(1/2)n(n-1)}\Delta^2_{(j)}(h'_{(j)})$. Equivalently, an element $g \in SU_{p,q}$ is regular if the eigenvalues of g are all distinct. Regular elements of $SU_{p,q}$ also have the property that all their principal minors are non-zero. The set $SU'_{p,q}$ of all regular elements of $SU_{p,q}$ is open and dense in $SU_{p,q}$ and its complement, the set of all singular elements of $SU_{p,q}$ is of measure zero with respect to the invariant Haar measure of $SU_{p,q}$ [26].

Now, define the mappings $\epsilon_{(j)\mathrm{R}}$ on H_j by

$$\begin{aligned}\epsilon_{(j)\mathrm{R}}(h_{(j)}) &= \mathrm{sgn}(\Delta_{(j)\mathrm{R}}(h_{(j)})) = \mathrm{sgn}\left(\prod_{k=1}^{j}(1-\mathrm{e}^{-2t_{(j)k}})\right)\\ &= \mathrm{sgn}\left(\prod_{k=1}^{j}(\sinh t_{(j)k})\right) = \mathrm{sgn}\left(\prod_{k=1}^{j}t_{(j)k}\right),\end{aligned} \tag{4.9}$$

where

$$\mathrm{sgn}(t) = \begin{cases} 1, & t > 0,\\ -1, & t < 0,\\ 0, & t = 0.\end{cases}$$

Now, where $w \in W(SU_{p,q}, H_j)$, and $h'_{(j)} \in H'_{(j)}$, define $\epsilon'_R(w)$ and $\epsilon_R(w)$ by

$$\begin{aligned}&\epsilon_{(j)\mathrm{R}}(wh'_{(j)}) = \epsilon'_R(w)\epsilon_{(j)\mathrm{R}}(h'_{(j)}),\\ &\epsilon_{(j)\mathrm{R}}(wh'_{(j)})\Delta_{(j)}(wj'_{(j)}) = \epsilon_R(w)\epsilon_{(j)\mathrm{R}}(h'_{(j)})\Delta_{(j)}(h'_{(j)}).\end{aligned} \tag{4.10}$$

A function f_j on H_j is said to be *skew symmetric* (*symmetric*) under S_{p-j} and under S_{q-j}, *symmetric* (*symmetric*) under S_j, and f_j is said to be *even* (*odd*) under $\mathbf{P}(j)$ if f_j satisfies

$$f_j(wh_{(j)}) = \epsilon_R(w) f_j(h_{(j)}) \quad (f_j(wh_{(j)}) = \epsilon'_R(w) f_j(h_{(j)})). \tag{4.11}$$

Let us now define [7] $SU''_{p,q} = \cup_{j=0}^{p}\{gh''_{(j)}g^{-1} : g \in SU_{p,q},\ h''_{(j)} \in H''_{(j)}\}$, where

$$H''_{(j)} = \left\{ h_{(j)} \in H_j : \prod_{R^+_{(j)\mathrm{R}} \cup R^+_{(j)\mathrm{C}} \cup R^+_{(j)\mathrm{SI}}} (1 - \xi_{(j)kl}(h^{-1}_{(j)})) \neq 0 \right\},$$

as the set of *quasi-regular* elements in $SU_{p,q}$. Then the set $SU''_{p,q}$ is an open dense subset of $SU_{p,q}$ and it is clear that one has $SU'_{p,q} \subset SU''_{p,q}$.

5. Invariant differential operators

The group $SU_{p,q}$, as a Lie group, can be seen as an analytic manifold of real dimension $(n^2 - 1)$, $n = p + q$. Let $C^\infty(SU_{p,q})$ denote the space of all infinitely differentiable, complex-valued functions on $SU_{p,q}$. The space $C^\infty(SU_{p,q})$ is a Fréchet space, and forms an algebra over $\mathbb{C}$ with pointwise linear operations. For each compact subset $K \subset SU_{p,q}$, let $C^\infty_K(SU_{p,q})$ denote a subspace of $C^\infty(SU_{p,q})$ equipped with the topology induced by $C^\infty(SU_{p,q})$. $C^\infty_K(SU_{p,q})$ is a closed subspace of $C^\infty(SU_{p,q})$, hence a Fréchet space. Let $C^\infty_\mathrm{c}(SU_{p,q})$ denote the subalgebra of $C^\infty(SU_{p,q})$ consisting of all functions on $SU_{p,q}$ with compact support. The space $C^\infty_\mathrm{c}(SU_{p,q})$ is given the *inductive limit topology* of the spaces $C^\infty_K(SU_{p,q})$. With this topology on $C^\infty_\mathrm{c}(SU_{p,q})$, the continuous linear functionals on $C^\infty_\mathrm{c}(SU_{p,q})$ are indeed the distributions on $SU_{p,q}$. According to a criterion by Peetre [9,17], a linear transformation $D : C^\infty_\mathrm{c}(SU_{p,q}) \to C^\infty_\mathrm{c}(SU_{p,q})$ is said to be a differential operator on $SU_{p,q}$ if it satisfies the condition $\mathrm{supp}(Df) \subset \mathrm{supp}(f)\,\forall f \in C^\infty_\mathrm{c}(SU_{p,q})$. The differential operator D on the analytic manifold $SU_{p,q}$ is said to be analytic if Df is analytic at a point $x \in SU_{p,q}$ whenever f is analytic at x. Let $\mathbf{E}(SU_{p,q})$ denote the set of all differential operators on $SU_{p,q}$. $\mathbf{E}(SU_{p,q})$ forms a subalgebra of $\mathrm{End}(C^\infty_\mathrm{c}(SU_{p,q}))$, the algebra of endomorphisms of $C^\infty_\mathrm{c}(SU_{p,q})$. Let $\mathbf{D}(SU_{p,q})$ denote the set of all left-invariant differential operators on $SU_{p,q}$. $\mathbf{D}(SU_{p,q})$ forms a subalgebra of $\mathbf{E}(SU_{p,q})$. The set of all *bi-invariant* differential operators on $SU_{p,q}$ is the set of all elements of $\mathbf{D}(SU_{p,q})$ which are also right-invariant.

A basis of $\mathbf{D}(SU_{p,q})$ can be obtained by introducing one-parameter subgroups of $SU_{p,q}$. Let $\tilde{X}$ denote the unique left-invariant vector field on $SU_{p,q}$ induced by $X \in \mathfrak{su}_{p,q}$ such that, for $f \in C^\infty_\mathrm{c}(SU_{p,q})$, $(\tilde{X}f)(g) = (\mathrm{d}/\mathrm{d}t) f(x\,\mathrm{e}^{tX})|_{t=0}$, $x \in SU_{p,q}$. $\tilde{X}$ defines a left-invariant differential operator on $SU_{p,q}$ which takes the form $\tilde{X} = \sum_{k=1}^{n^2-1} a_k(\partial/\partial x_k)$, where $a_k \in$

$C^\infty(SU_{p,q})$ and $(x_1, \ldots, x_{n^2-1})$ are coordinates of $x \in SU_{p,q}$. Where $\{X_1, \ldots, X_{n^2-1}\}$ is a basis for $\mathfrak{su}_{p,q}$, the set of monomials $\{\tilde{X}_1^{k_1} \cdots \tilde{X}_{n^2-1}^{k_{n^2-1}} : k_i \in \mathbb{N} \cup \{0\}\}$, forms a basis for $\mathbf{D}(SU_{p,q})$.

6. Universal enveloping algebra of $SU_{p,q}$

Recalling that $\mathfrak{sl}_n(\mathbb{C})$ is the complexification of $\mathfrak{su}_{p,q}$, let $\mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}))$ denote the *universal enveloping algebra* of $\mathfrak{su}_{p,q}$. This algebra is defined as the factor algebra $(\mathfrak{sl}_n(\mathbb{C}))^{\otimes}/J_u$, where $(\mathfrak{sl}_n(\mathbb{C}))^{\otimes}$ is the tensor algebra over $\mathfrak{sl}_n(\mathbb{C})$ (considered as a vector space), given by

$$(\mathfrak{sl}_n(\mathbb{C}))^{\otimes} = \mathbb{C} \oplus \sum_{k\geq 1}^{\oplus} \underbrace{\mathfrak{sl}_n(\mathbb{C}) \otimes \cdots \otimes \mathfrak{sl}_n(\mathbb{C})}_{k \text{ times}} := (\mathfrak{sl}_n(\mathbb{C}))_0^{\otimes} \oplus (\mathfrak{sl}_n(\mathbb{C}))_+^{\otimes},$$

and J_u is the two-sided ideal in $(\mathfrak{sl}_n(\mathbb{C}))^{\otimes}$ generated by all tensor elements of the form $X \otimes Y - Y \otimes X - [X, Y]$, where $X, Y \in \mathfrak{sl}_n(\mathbb{C})$.

Let, in the following diagram, σ_1 be the canonical injection and σ_2 be the natural homomorphism

$$\mathfrak{sl}_n(\mathbb{C}) \xrightarrow{\sigma_1} \mathfrak{sl}_n(\mathbb{C})^{\otimes} \xrightarrow{\sigma_2} \mathfrak{U}(\mathfrak{sl}_n(\mathbb{C})).$$

The composite mapping $\sigma = \sigma_2 \circ \sigma_1$ is a Lie homomorphism of $\mathfrak{sl}_n(\mathbb{C})$ into $\mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}))$, i.e., for all $X, Y \in \mathfrak{sl}_n(\mathbb{C})$ one has $\sigma([X, Y]) = \sigma(X)\sigma(Y) - \sigma(Y)\sigma(X)$, where $\sigma(X)\sigma(Y) = (X \otimes Y) + J_u = \sigma_2(X \otimes Y)$. The algebra $\mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}))$ over $\mathbb{C}$ is an associative algebra with respect to the usual coset multiplication. One can show that the canonical mapping σ is injective and hence one may identify every element of $\mathfrak{sl}_n(\mathbb{C})$ (and consequently every element of $\mathfrak{su}_{p,q}$) with its canonical image in $\mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}))$.

Let $\{X_1, \ldots, X_{n^2-1}\}$ be a basis of $\mathfrak{sl}_n(\mathbb{C})$. The elements of this basis satisfy $[X_i, X_j] = \sum_{k=1}^{n^2-1} c_{ij}^k X_k$, where $c_{ij}^k \in \mathbb{R}$ are the *structure constants*. In terms of this basis, a basis of $(\mathfrak{sl}_n(\mathbb{C}))^{\otimes}$ is given by

$$\{1, X_{i_1} \otimes \cdots \otimes X_{i_k} : 1 \leq i_1, \ldots, i_k \leq n^2 - 1,\ k \in \mathbb{N}\}.$$

The *Poincaré–Birkhoff–Witt basis* of $\mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}))$ is given by

$$\{(\sigma(X_1))^{k_1} \cdots (\sigma(X_{n^2-1}))^{k_{n^2-1}} : k_i \in \mathbb{N} \cup \{0\}\}, \tag{6.1}$$

where $\sigma(X_i) = X_i + J_u$ and satisfies $[\sigma(X_i), \sigma(X_j)] = \sum_{k=1}^{n^2-1} c_{ij}^k \sigma(X_k)$. Clearly, with this basis for $\mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}))$ and the basis for $\mathbf{D}(SU_{p,q})$ (see Section 5), the isomorphism

$$\mathfrak{U}(\mathfrak{sl}_n(\mathbb{C})) \cong \mathbf{D}(SU_{p,q}) \tag{6.2}$$

can be immediately seen.

Now, consider a point x in $SU_{p,q}$, and let $\mathfrak{X} \in \mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}))$. One can show that if $\mathfrak{X} f(x) = 0$ for all $f \in C_c^\infty(SU_{p,q})$, then $\mathfrak{X} = 0$. Furthermore, it is known that if D is a differential operator on $SU_{p,q}$, then there exists exactly one element $\mathfrak{X}_x \in \mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}))$ such that

$(Df)(x) = (\mathfrak{X}_x f)(x)\,\forall f \in C_c^\infty(SU_{p,q})$. Such an element $\mathfrak{X}_x$ is called the *local expression of* D at x [7].

Let $\mathfrak{Z}$ denote the center of $\mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}))$. By the isomorphism (6.2), one has

$$\mathfrak{Z} \cong \mathscr{Z}(\mathbf{D}(SU_{p,q})), \tag{6.3}$$

where

$$\mathfrak{Z} = \{\mathfrak{X} \in \mathfrak{U}(\mathfrak{sl}_n(\mathbb{C})) : [X, \mathfrak{X}] = 0\,\forall X \in \mathfrak{su}_{p,q}\}. \tag{6.4}$$

7. The Harish-Chandra homomorphisms

Let $\mathfrak{h}_j^{\mathbf{c}},\ j = 0, \ldots, p$ denote the complexification of the Cartan subalgebra $\mathfrak{h}_j$ of $\mathfrak{su}_{p,q}$. Let $\mathfrak{U}(\mathfrak{h}_j^{\mathbf{c}})$ denote the universal enveloping algebra generated by 1 and $\mathfrak{h}_j^{\mathbf{c}}$. Clearly $\mathfrak{U}(\mathfrak{h}_j^{\mathbf{c}})$ is a subalgebra of $\mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}))$. Because each $\mathfrak{h}_j^{\mathbf{c}}$ is a conjugate form of the Cartan subalgebra $\mathfrak{h}_{\mathrm{d}}$ of $\mathfrak{sl}_n(\mathbb{C})$, the $(p+1)$ enveloping algebras $\mathfrak{U}(\mathfrak{h}_j^{\mathbf{c}})$ are also conjugate to $\mathfrak{U}(\mathfrak{h}_{\mathrm{d}})$. Consequently, one considers only $\mathfrak{U}(\mathfrak{h}_{\mathrm{d}})$. A basis of $\mathfrak{U}(\mathfrak{h}_{\mathrm{d}})$ is given by

$$\{(\sigma(X_1))^{k_1} \cdots (\sigma(X_n))^{k_n} : k_i \in \mathbb{N} \cup \{0\}\}, \tag{7.1}$$

where $\{X_1, \ldots, X_n;\ X_1 + \cdots + X_n = 0\}$ is a basis of $\mathfrak{h}_{\mathrm{d}}$. The Poincaré–Birkhoff–Witt basis for $\mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}))$ can be given by the set of monomials, for $q_k, r_k, p_k \in \mathbb{N} \cup \{0\}$,

$$\prod_{\alpha_k \in R^+} (\sigma(E_{-\alpha_k}))^{q_k} \prod_{k=1}^{n} (\sigma(X_{(j)k}))^{r_k} \prod_{\alpha_k \in R^+} (\sigma(E_{\alpha_k}))^{p_k}, \tag{7.2}$$

which follows from the triangular decomposition $\mathfrak{sl}_n(\mathbb{C})$. Since $\mathfrak{h}_{\mathrm{d}}$ is abelian, $\mathfrak{U}(\mathfrak{h}_{\mathrm{d}})$ coincides with the symmetric algebra $\mathfrak{S}(\mathfrak{h}_{\mathrm{d}}) = (\mathfrak{h}_{\mathrm{d}})^{\otimes}/J_{\mathrm{s}}$, where J_{s} is a two-sided ideal with elements of the form, $X \otimes Y - Y \otimes X\,\forall X, Y \in \mathfrak{h}_{\mathrm{d}}$.

Let $\sigma : \mathfrak{sl}_n(\mathbb{C}) \to \mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}))$ be the Lie homomorphism, and let $\sigma_{\mathrm{s}} : \mathfrak{sl}_n(\mathbb{C}) \to \mathfrak{S}(\mathfrak{sl}_n(\mathbb{C}))$ be the canonical mapping. Then, on viewing these algebras as vector spaces, there exists a unique linear isomorphism, called *symmetrization*, defined by $\lambda : \mathfrak{S}(\mathfrak{sl}_n(\mathbb{C})) \to \mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}))$, such that $\sigma = \lambda \circ \sigma_{\mathrm{s}}$. It can be shown [15,17] that the image of the $\mathrm{Ad}(SL_n(\mathbb{C}))$-invariant subset of $\mathfrak{S}(\mathfrak{sl}_n(\mathbb{C}))$ under the symmetrization mapping λ is equal to the center $\mathfrak{Z}$. In terms of the basis (7.2), one has the decomposition

$$\mathfrak{U}(\mathfrak{sl}_n(\mathbb{C})) = \mathfrak{U}(\mathfrak{h}_{\mathrm{d}}) \oplus [\sigma(\mathfrak{n}^-)\mathfrak{U}(\mathfrak{sl}_n(\mathbb{C})) + \mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}))\sigma(\mathfrak{n}^+)], \tag{7.3}$$

which gives raise to the following two conditions:

$$\mathfrak{U}(\mathfrak{h}_{\mathrm{d}}) \cap \sum_{\alpha \in R^+} \mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}))\sigma(E_\alpha) = \{0\}, \tag{7.4a}$$

$$\mathfrak{Z} \subseteq \mathfrak{U}(\mathfrak{h}_{\mathrm{d}}) \oplus \sum_{\alpha \in R^+} \mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}))\sigma(E_\alpha). \tag{7.4b}$$

Let γ' be the projection of $\mathfrak{U}(\mathfrak{h}_\mathrm{d}) \oplus \sum_{\alpha\in R^+}\mathfrak{U}(\mathfrak{s}l_n(\mathbb{C}))\sigma(E_\alpha)$ onto the $\mathfrak{U}(\mathfrak{h}_\mathrm{d})$ component. Clearly, one can see that

$$\gamma'\left(\sum_{\alpha\in R^+}\mathfrak{U}(\mathfrak{s}l_n(\mathbb{C}))\sigma(E_\alpha)\right) = \{0\}. \tag{7.5}$$

The mapping γ' is an algebra homomorphism of $\mathfrak{Z}$ into $\mathfrak{U}(\mathfrak{h}_\mathrm{d})$ [5]. Following Harish-Chandra [7], we define $\eta \in \mathrm{Aut}(\mathfrak{U}(\mathfrak{h}_\mathrm{d}))$ such that $\eta(1) = 1$ and for $X_1, X_2 \in \mathfrak{h}_\mathrm{d}$,

$$\eta(\sigma_1(X_1)\otimes\sigma_1(X_2) + J_\mathrm{s}) = (\sigma_1(X_1)\otimes\sigma_1(X_2) + \delta(X_1)\sigma_1(X_2)$$
$$+\delta(X_2)\sigma_1(X_1) + \delta(X_1)\delta(X_2)) + J_\mathrm{s}, \quad \text{and}$$
$$\eta(1 + J_\mathrm{s}) = (1 + J_\mathrm{s}).$$

This reduces, when $X_1 = X,\ X_2 = 0$, to $\eta(\sigma_1(X) + J_\mathrm{s}) = (\sigma_1(X) - \delta(X)) + J_\mathrm{s}$, where δ is given by (4.6) ($\delta(X) \in \mathbb{C}$). One can see from the definition of γ' and the definition of η, that η is multiplicative on the center $\mathfrak{Z}$. Now define the mapping $\gamma = \eta \circ \gamma' : \mathfrak{Z} \to \mathfrak{U}(\mathfrak{h}_\mathrm{d})$.

Lemma 7.1. $\gamma_j \in \mathrm{Hom}(\mathfrak{Z}, \mathfrak{U}(\mathfrak{h}_j^\mathfrak{c}))\ \forall j = 0, 1, \ldots, p$.

Proof. For $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathfrak{Z}$, the element

$$\mathfrak{z}_1\mathfrak{z}_2 - \gamma'(\mathfrak{z}_1)\gamma'(\mathfrak{z}_2) = \mathfrak{z}_1(\mathfrak{z}_2 - \gamma'(\mathfrak{z}_2)) + \gamma'(\mathfrak{z}_2)(\mathfrak{z}_1 - \gamma'(\mathfrak{z}_1))$$

is in $\sum_{\alpha\in R^+}\mathfrak{U}(\mathfrak{s}l_n(\mathbb{C}))\sigma(E_\alpha)$. Hence from (7.5), one has $\gamma'(\mathfrak{z}_1\mathfrak{z}_2) - \gamma'(\mathfrak{z}_1)\gamma'(\mathfrak{z}_2) = 0$. That is, $\gamma'(\mathfrak{z}_1\mathfrak{z}_2) = \gamma'(\mathfrak{z}_1)\gamma'(\mathfrak{z}_2)$, and thus γ' is multiplicative on $\mathfrak{Z}$. Since η is multiplicative one finds that γ also multiplicative. As the composition of two algebraic homomorphisms the mapping γ is also an algebraic homomorphism, and is called the *Harish-Chandra homomorphism*. □

An element $\mathfrak{X} \in \mathfrak{U}(\mathfrak{h}_\mathrm{d})$ is invariant under the Weyl group $W(\mathfrak{s}l_n(\mathbb{C}), \mathfrak{h}_\mathrm{d})$ if $\mathfrak{X} = w\mathfrak{X}w^{-1}$ for all $w \in W(\mathfrak{s}l_n(\mathbb{C}), \mathfrak{h}_\mathrm{d})$. The set of all Weyl invariant elements form a subalgebra $\mathfrak{I}(\mathfrak{h}_\mathrm{d})$ of $\mathfrak{U}(\mathfrak{h}_\mathrm{d})$. One can prove [18] that $\mathrm{Im}(\gamma) \subseteq \mathfrak{I}(\mathfrak{h}_\mathrm{d})$ and that the mapping $\gamma : \mathfrak{Z} \to \mathfrak{I}(\mathfrak{h}_\mathrm{d})$ is bijective. Thus one has $(p+1)$ number of isomorphisms $\gamma_j : \mathfrak{Z} \to \mathfrak{I}(\mathfrak{h}_j^\mathfrak{c})$.

Now, let $\tau \in \mathrm{Aut}(\mathfrak{s}l_n(\mathbb{C}))$ be defined such that $\tau(1) = 1$ and $\tau(\mathfrak{h}_j^\mathfrak{c}) = \mathfrak{h}_j^\mathfrak{c}$. That is, $\mathfrak{h}_j^\mathfrak{c}$ is τ-invariant. Since $\eta \in \mathrm{Aut}(\mathfrak{U}(\mathfrak{h}_\mathrm{d}))$ and since, as pointed out in Section 3, the root system of $(\mathfrak{s}l_n(\mathbb{C}), \mathfrak{h}_\mathrm{d})$ parallels that of $(\mathfrak{s}l_n(\mathbb{C}), \mathfrak{h}_j^\mathfrak{c})$, one finds that the composition $\tau \circ \eta$ is commutative. From the definitions of γ_j' and γ_j and by using the τ-invariance of $\mathfrak{h}_j^\mathfrak{c}$ and the isomorphism $\mathfrak{Z} \to \mathfrak{I}(\mathfrak{h}_j^\mathfrak{c})$ it can be shown that, for $\mathfrak{z} \in \mathfrak{Z}$, $\gamma_j'(\tau\mathfrak{z}\tau^{-1}) = \tau\gamma_j'(\mathfrak{z})\tau^{-1}$, implying that $\gamma_j(\tau\mathfrak{z}\tau^{-1}) = \tau\gamma_j(\mathfrak{z})\tau^{-1}$.

Because the rank of $SU_{p,q}$ is equal to $(n-1)$, the center $\mathfrak{Z}$ is generated by $(n-1)$ basis elements. These basis elements can be chosen to be polynomials in the basis elements of $\mathfrak{s}u_{p,q}$. These $(n-1)$ elements are called the *Casimir elements*, denoted by $\mathfrak{C}_\kappa,\ \kappa = 2, \ldots, n$. Writing the set of basis elements of $\mathfrak{s}u_{p,q}$ as $\{X_{ik} : 1 \le i,\ k \le n,\ X_{11} + \cdots + X_{nn} = 0\}$, the Casimir elements may be defined as

$$\mathfrak{C}_\kappa = \sum_{i_1,\dots,i_\kappa=1}^{n} \left(\prod_{k=1}^{\kappa-1} \sigma(X_{i_k i_{k+1}}) \right) \sigma(X_{i_\kappa i_1}), \tag{7.6}$$

satisfying $[\mathfrak{C}_\kappa, X_{ik}] = 0,\ i, k = 1, \dots, n$. To determine the action of the projection γ' on $\mathfrak{C}_\kappa$, one expands each $\mathfrak{C}_\kappa$ and uses the commutation relations to move all the $X_{ik},\ i < k$, toward the right. That is, writing $\mathfrak{C}_\kappa$ in the form $\mathfrak{U}(\mathfrak{h}_\mathrm{d}) \oplus \sum_{\alpha \in R^+_{(j)}} \mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}))\sigma(E_\alpha)$. Then, by (7.5), the resulting expressions become polynomials in only $X_{kk} := X_k,\ k = 1, \dots, n$, which are the elements of $\mathfrak{U}(\mathfrak{h}_\mathrm{d})$. That is,

$$\gamma'(\mathfrak{C}_\kappa) = \sum_{i=1}^{n} (\sigma(X_i))^\kappa + \sum_{k=1}^{\kappa-1} \sum_{l=1}^{n} a_{kl} (\sigma(X_l))^k,$$

where $a_{kl} \in \mathbb{Q}$ can be uniquely determined by the commutation relations on X_{ik}.

In order to determine the action of the Harish-Chandra isomorphism $\gamma = \eta \circ \gamma'$, one also needs to know the action of η, which requires the action of δ. δ is one-half of the sum of all positive roots of $(\mathfrak{sl}_n(\mathbb{C}), \mathfrak{h}_\mathrm{d})$ and is given by (4.6). That is, $\delta(X_k) = \frac{1}{2}(n - 2k + 1)$. The action of the automorphism η on $\sigma(X_k)$ is now given by $\eta(\sigma(X_k)) = \sigma(X_k) - \frac{1}{2}(n - 2k + 1)$. Hence the action of the Harish-Chandra isomorphism γ on $\mathfrak{C}_\kappa$ is given by

$$\gamma(\mathfrak{C}_\kappa) = \sum_{k=1}^{n} \left[\sigma(X_k) - \frac{1}{2}(n - 2k + 1) \right]^\kappa + \sum_{k=1}^{\kappa-1} \sum_{l=1}^{n} a_{kl} \left[\sigma(X_l) - \frac{1}{2}(n - 2l + 1) \right]^k. \tag{7.7}$$

Let, as in Section 4, $H'_{(j)} = H_j \cap SU'_{p,q}$ denote the set of regular elements in H_j. $H'_{(j)}$ may be seen as an open submanifold of $SU_{p,q}$. A measure on a manifold [26] is said to be *equivalent to Lebesgue measure* if on each coordinate neighborhood, it is a multiple of Lebesgue measure by a nowhere vanishing C^∞ function. Then, one has the following special cases of the theorems from [9,15].

Theorem 7.2 (Harish-Chandra [5–14]). *Where π is a projection of $SU_{p,q}$ onto $H'_{(j)}$ and where $\mathrm{d}\mu(g)$ and $\mathrm{d}\mu(h_{(j)})$ are equivalent to Lebesgue measures on $SU_{p,q}$ and $H'_{(j)}$, respectively, then there exists a unique function $f_\alpha \in C^\infty_\mathrm{c}(H'_{(j)})$ for each $\alpha \in C^\infty_\mathrm{c}(SU_{p,q})$ such that*

$$\int_{SU_{p,q}} (F \circ \pi)(g)\alpha(g)\, \mathrm{d}\mu(g) = \int_{H'_{(j)}} F(h_{(j)}) f_\alpha(h_{(j)})\, \mathrm{d}\mu(h_{(j)}) \quad \forall F \in C^\infty_\mathrm{c}(H'_{(j)}).$$

Furthermore, $\alpha \mapsto f_\alpha$ is a continuous mapping of $C^\infty_\mathrm{c}(SU_{p,q})$ onto $C^\infty_\mathrm{c}(H'_{(j)})$, and $\mathrm{supp}(f_\alpha) \subseteq \pi(\mathrm{supp}(\alpha))$.

Theorem 7.3 (Helgason [15–18]). *If D is a differential operator on $SU_{p,q}$, then there exists a unique differential operator $\Delta(D)$ on $H'_{(j)}$, called the radial part of D, such that $(Df)|_{H'_{(j)}} = \Delta(D) f|_{H'_{(j)}}$ for each locally invariant C^∞ function f on $SU_{p,q}$.*

From the above two theorems, one obtains the following expression for the radial part of the differential operator $\mathfrak{z} \in \mathfrak{Z}$ in terms of $\gamma(\mathfrak{z}) \in \mathfrak{I}(\mathfrak{h}_{\mathrm{d}})$:

$$(\mathfrak{z} f)|_{H'_{(j)}}(h_{(j)})=((\Delta_{(j)}(h_{(j)}))^{-1}\gamma(\mathfrak{z}) \circ \Delta_{(j)}(h_{(j)})) f|_{H'_{(j)}}(h_{(j)}), \quad h_{(j)} \in H'_{(j)}, \tag{7.8}$$

where $\circ$ denotes composition of differential operators, and $\Delta_j(h_{(j)})$ is the density function given in (4.8). The composite operator

$$(\Delta_{(j)}(h_{(j)}))^{-1}\gamma(\mathfrak{z}) \circ \Delta_{(j)}(h_{(j)}) \tag{7.9}$$

is well defined and is the radial part of $\mathfrak{z}$ on $H'_{(j)}$.

One can express every differential operator $\gamma(\mathfrak{z}) \in \mathfrak{I}(\mathfrak{h}_{\mathrm{d}})$ in terms of differential operators $h_{(j)k}(\partial/\partial h_{(j)k})$ (see (3.8)). These differential operators are given by

$$h_{(j)k}\frac{\partial}{\partial h_{(j)k}} = \begin{cases} \dfrac{1}{\iota}\dfrac{\partial}{\partial \phi_{(j)k}}, & k=1,\dots,p-j, \\[2ex] \dfrac{\partial}{\partial z_{(j)p-k+1}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial t_{(j)p-k+1}} + \dfrac{1}{\iota}\dfrac{\partial}{\partial \theta_{(j)p-k+1}}\right), & k=p-j+1,\dots,p, \\[2ex] -\dfrac{\partial}{\partial \bar{z}_{(j)k-p}} = \dfrac{1}{2}\left(-\dfrac{\partial}{\partial t_{(j)k-p}} + \dfrac{1}{\iota}\dfrac{\partial}{\partial \theta_{(j)k-p}}\right), & k=p+1,\dots,p+j, \\[2ex] \dfrac{1}{\iota}\dfrac{\partial}{\partial \psi_{(j)n-k+1}}, & k=p+j+1,\dots,p+q=n. \end{cases} \tag{7.10}$$

The set of these operators forms a basis for $\mathfrak{I}(\mathfrak{h}_{\mathrm{d}})$. As we see in Section 8, for a given representation of $SU_{p,q}$, the eigenvalues of these basis elements are given by the parameters $u_{(j)} = (u_{(j)1},\dots,u_{(j)n})$. The n-tuple $u_{(j)}$ is the *highest weight* of the representation of $SU_{p,q}$ induced by H_j.

8. Character groups and representation parameters

The *character group* of the dense subset $H'_{(j)},\ 0 \le j \le p$ of the Cartan subgroup H_j is given by the set of mappings $\xi_{(j)\alpha_{u_j}} \in \mathrm{Hom}(H'_{(j)}, \mathbb{C})$. We denote this group by $H'^{*}_{(j)}$. From (3.5) and (3.8), one obtains for $h_{(j)} \in H'_{(j)}$,

$$\begin{aligned} \xi_{(j)\alpha_{u_j}}(h_{(j)}) &= \xi_{(j)\alpha_{u_j}}(\mathrm{diag}[h_{(j)1},\dots,h_{(j)n}]) = \prod_{k=1}^{n}(h_{(j)k})^{u_{(j)k}} \\ &= \xi_{(j)\alpha_{u_j}}(\mathrm{diag}[\mathrm{e}^{\iota\phi_1},\dots,\mathrm{e}^{\iota\phi_{p-j}};\mathrm{e}^{z_j},\dots,\mathrm{e}^{z_1};\mathrm{e}^{-\bar{z}_1},\dots,\mathrm{e}^{-\bar{z}_j};\mathrm{e}^{\iota\psi_{q-j}},\dots,\mathrm{e}^{\iota\psi_1}]) \\ &= \left(\prod_{k=1}^{p-j}(\mathrm{e}^{\iota\phi_k})^{u_{(j)k}}\right)\left(\prod_{k=p-j+1}^{p}(\mathrm{e}^{z_{p-k+1}})^{u_{(j)k}}\prod_{k=p+1}^{p+j}(z^{-\bar{z}_{k-p}})^{u_{(j)k}}\right) \\ &\quad\times\left(\prod_{k=p+j+1}^{n}(\mathrm{e}^{\iota\psi_{n-k+1}})^{u_{(j)k}}\right), \end{aligned}$$

where $u_{(j)k} \in \mathbb{Z}$ for $1 \le k \le p-j$ and $p+j+1 \le k \le n$, and $u_{(j)k} \in \mathbb{C}$ for $p-j+1 \le k \le p+j$. The n-tuple $(u_{(j)1}, \dots, u_{(j)n})$ is called the *signature* of the character $\xi_{(j)\alpha_{u_j}}$. Clearly one has that

$$\left(h_{(j)k}\frac{\partial}{\partial h_{(j)k}}\xi_{(j)\alpha_{u_j}}\right)(h_{(j)}) = u_{(j)k}\xi_{(j)\alpha_{u_j}}(h_{(j)}). \tag{8.1}$$

A character $\xi_{(j)\alpha_{u_j}} \in H'^{*}_{(j)}$ is *regular* since, for every non-unit element $w \in W(SU_{p,q}, H_j)$, $w(\alpha_u) \neq \alpha_u$.

By using the unimodularity condition given by $h_{(j)n} = (h_{(j)1}h_{(j)2}\cdots h_{(j)n-1})^{-1}$, i.e.,

$$\psi_1 = -\sum_{k=1}^{p-j}\phi_k - 2\sum_{k=1}^{j}\theta_k - \sum_{k=2}^{q-j}\psi_k,$$

one obtains, for $u'_{(j)k} = u_{(j)k} - u_{(j)n}$,

$$\xi_{(j)\alpha_{u_j}}(h_{(j)}) = \left(\prod_{k=1}^{p-j}(\mathrm{e}^{\iota\phi_k})^{u'_{(j)k}}\right)\left(\prod_{k=p-j+1}^{p}(\mathrm{e}^{z_{p-k+1}})^{u'_{(j)k}}\prod_{k=p+1}^{p+j}(\mathrm{e}^{-\bar{z}_{k-p}})^{u'_{(j)k}}\right) \times \left(\prod_{k=p+j+1}^{n-1}(\mathrm{e}^{\iota\psi_{n-k+1}})^{u'_{(j)k}}\right).$$

In fact, there are

$$\binom{n}{1}$$

ways in which one can introduce the unimodularity condition.

It is known that the representations of $SU_{p,q}$ induced by the Cartan subgroups H_j are parameterized by the signatures of the characters in the respective character groups $H'^{*}_{(j)}$. Choosing any of the

$$\binom{n}{1}$$

ways to introduce the unimodularity condition will result in equivalent representations. For unitary representations of $SU_{p,q}$, one may choose the parameterization

$$u'_{(j)k} = u_{(j)k} - u_{(j)n} = \begin{cases} m_{(j)k}, & 1 \le k \le p-j, \\ \sigma_{(j)k} := \frac{1}{2}(\lambda_{(j)k} + \iota r_{(j)k}), & p-j+1 \le k \le p, \\ \sigma_{(j)k} := \frac{1}{2}(\lambda_{(j)2p-k+1} - \iota r_{(j)2p-k+1}), & p+1 \le k \le p+j, \\ m_{(j)k}, & p+j+1 \le k \le n-1. \end{cases} \tag{8.2}$$

Consequently, one has

$$\xi_{(j)\alpha_{u_j}}(h_{(j)}) = \left(\prod_{k=1}^{p-j}(\mathrm{e}^{\iota\phi_k})^{m_{(j)k}}\right)\left(\prod_{k=p-j+1}^{p}\left(\frac{\mathrm{e}^{z_{p-k+1}}}{|\mathrm{e}^{z_{p-k+1}}|}\right)^{\lambda_{(j)k}}|\mathrm{e}^{z_{p-k+1}}|^{\iota r_{(j)k}}\right) \times\left(\prod_{k=p+j+1}^{n-1}(\mathrm{e}^{\iota\psi_{n-k+1}})^{m_{(j)k}}\right), \tag{8.3}$$

where $m_{(j)k} \in \mathbb{Z}$, $\lambda_{(j)k}, r_{(j)k} \in \mathbb{R}$, and $\mathrm{e}^{z(1/2)(m+\iota r)}\,\mathrm{e}^{-\bar{z}(1/2)(m-\iota r)} = (\mathrm{e}^z/|\mathrm{e}^z|)^m|\mathrm{e}^z|^{\iota r}$. For irreducible unitary representations of $SU_{p,q}$, one has the Weyl condition [30]: $m_{(j)1} \geq m_{(j)2} \geq \cdots \geq m_{(j)p-j} \geq m_{p+j+1} \geq m_{p+j+2} \geq \cdots \geq m_{(j)n-1}$.

Harish-Chandra [11,12] proved that a semisimple Lie group has a discrete series of representations if and only if its rank is equal to the rank of its maximal compact subgroup or, equivalently, if and only if it has a compact Cartan subgroup. The group $SU_{p,q}$ has one compact Cartan subgroup $H_0 \subset K = S(U(p) \otimes U(q))$. Also, the rank of $SU_{p,q} = (n-1) = \mathrm{rank}(K)$, hence $SU_{p,q}$ has a discrete series of representations.

Let $R^+_{(0)\mathrm{c}}$ be a fixed system of compact positive roots given by (4.3). There exist exactly

$$\binom{n}{p}$$

systems of positive roots, denoted by

$$R^+_k, \quad 1 \leq k \leq \binom{n}{p},$$

containing $R^+_{(0)\mathrm{c}}$. These systems are given by $R^+_k = w_k R^+_{(0)}$, where $R^+_{(0)}$ is given by (4.2) and the Weyl reflections $w_k \in W(\mathfrak{sl}_n(\mathbb{C}))$ are products of transpositions $s_i = (i, i+1)$, $i = 1, \dots, p$. This means that there are

$$\binom{n}{p}$$

non-equivalent discrete representations for $SU_{p,q}$. For each of the

$$\binom{n}{p}$$

systems of positive roots R^+_k containing $R^+_{(0)\mathrm{c}}$, one can compute

$$\delta_k = \frac{1}{2}\sum_{\alpha\in R^+_k}\alpha,\ k = 1, 2, \dots, \binom{n}{p}.$$

Since each Weyl reflection s_i changes a non-compact negative root into a non-compact positive root, one may easily compute δ_k for each k. For example, when $w_1 = I$ and $w_2 = s_p$ one obtains, where $q - p = n - 2p$,

$$\delta_1 = \tfrac{1}{2}(n-1, n-3, \ldots, q-p+1, q-p-1, q-p-3, \ldots, -n+3, -n+1),$$
$$\delta_2 = \tfrac{1}{2}(n-1, n-3, \ldots, q-p-1, q-p+1, q-p-3, \ldots, -n+3, -n+1).$$

The only *holomorphic* discrete series representations of $SU_{p,q},\ p < q$, is obtained from δ_1. However, when $p = q$, there exists a system of positive roots, R_a^+, in which all p^2 non-compact negative roots enter as positive roots. In this case, one obtains

$$\delta_1 = \tfrac{1}{2}(n-1, n-3, \ldots, 3, 1, -1, -3, \ldots, -n+3, -n+1),$$
$$\delta_a = \tfrac{1}{2}(-1, -3, \ldots, -n+3, -n+1, n-1, n-3, \ldots, 3, 1).$$

Here, the only holomorphic series representations is obtained from δ_1, and the only *anti-holomorphic* series representations is obtained from δ_a.

One can introduce, for discrete series of representations, a different set of parameters, called *Harish-Chandra parameters* $\ell = (\ell_1, \ldots, \ell_n)$, or Blattner parameters $\flat = (\flat_1, \ldots, \flat_n)$. These parameters are related by $\flat = \ell - 2\delta_{\mathrm{c}} + \delta_k$, where $\delta_{\mathrm{c}} = \frac{1}{2}\sum_{\alpha \in R^+_{(0)\mathrm{c}}} \alpha = \frac{1}{2}(p-1, p-3, \ldots, -p+1, q-1, q-3, \ldots, -q+1)$. However, in the discussion which follows the parameters $\{u_{(j)k}\}_{k=1}^n$, as given in (8.2), are used since it is possible to use them generally for both discrete and continuous series of representations of $SU_{p,q}$.

9. Gårding space and representations of the Lie algebra $\mathfrak{su}_{p,q}$

Let $(\rho_j, \mathcal{H}_j)$ denote the representation of $SU_{p,q}$ induced by the Cartan subgroups $H_j,\ j = 0, 1, \ldots, p$. That is, $\rho_j \in \mathrm{Hom}(SU_{p,q}, \mathrm{Aut}(\mathcal{H}_j))$. Gårding [3] showed that every group representation $(\rho, \mathcal{H})$ defines a representation of its corresponding Lie algebra on a dense subspace of $\mathcal{H}$. Let $\mathcal{G}_j$ be a vector subspace of $\mathcal{H}_j$ spanned by all vectors of the form

$$\int_{SU_{p,q}} f(g)(\rho_j(g))(\psi)\,\mathrm{d}\mu(g) := \mathring{\psi}_f, \tag{9.1}$$

where $\mathrm{d}\mu(g)$ is the left-invariant Haar measure [26] on $SU_{p,q},\ \psi \in \mathcal{H}_j$ and $f \in C_{\mathrm{c}}^{\infty}(SU_{p,q})$. The space $\mathcal{G}_j$ is called the *Gårding subspace* of $\mathcal{H}_j$ (or *Gårding domain*) with respect to the group representation ρ_j. For every $X \in \mathfrak{su}_{p,q}$ one defines a linear operator $\varrho_j(X)$ of $\mathcal{G}_j$ into itself such that

$$\varrho_j(X)\mathring{\psi}_f = \left.\frac{\mathrm{d}}{\mathrm{d}t}(\rho_j(\mathrm{e}^{tX})\mathring{\psi}_f)\right|_{t=0} = \lim_{t\to 0}\left(\frac{\rho_j(\mathrm{e}^{tX}) - I}{t}\right)\mathring{\psi}_f, \quad t \in \mathbb{R}. \tag{9.2}$$

One calls the pair $(\varrho_j, \mathcal{G}_j)$, the *derived* or *differentiated* representation of the group $SU_{p,q}$. If $g_0 \in SU_{p,q}$, then

$$\begin{aligned}\rho_j(g_0)\mathring{\psi}_f &= \rho_j(g_0)\int_{SU_{p,q}} f(g)\rho_j(g)(\psi)\,\mathrm{d}\mu(g) = \int_{SU_{p,q}} f(g)\rho_j(g_0 g)(\psi)\,\mathrm{d}\mu(g)\\ &= \int_{SU_{p,q}} f(g_0^{-1}z)\rho_j(z)(\psi)\,\mathrm{d}\mu(z) = \int_{SU_{p,q}} f^{\tau_{g_0}}(z)\rho_j(z)(\psi)\,\mathrm{d}\mu(z) = \mathring{\psi}_{f^{\tau_{g_0}}},\end{aligned} \tag{9.3}$$

where τ_{g_0} is the left translation. One can easily prove [3] the following theorem.

Theorem 9.1. *Let $(\rho_j, \mathcal{H}_j)$ be a representation of $SU_{p,q}$. Then*

1. *The Gårding space $\mathcal{G}_j$ is dense in $\mathcal{H}_j$.*
2. *The Gårding space $\mathcal{G}_j$ is stable under $\varrho_j(X)$, where X is the generator of a one-parameter subgroup of $SU_{p,q}$.*
3. *The pair $(\varrho_j, \mathcal{G}_j)$ with $\varrho_j(X)\overset{\circ}{\psi}_f = \overset{\circ}{\psi}_{\tilde{X}(f)}\ \forall X \in \mathfrak{su}_{p,q}$ is a representation of $\mathfrak{su}_{p,q}$.*
4. *The operators $\iota\varrho_j(X)$, $\iota = \sqrt{-1}$, $X \in \mathfrak{su}_{p,q}$ are symmetric.*

Remark.

1. Since $\mathcal{G}_j$ is an invariant subspace, one can define the action of any element $\mathfrak{X} \in \mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}))$ by $\varrho_j(\mathfrak{X})\overset{\circ}{\psi}_f = \overset{\circ}{\psi}_{\tilde{\mathfrak{X}}(f)}$, where $\tilde{\mathfrak{X}}$ is the *left* differential operator on $C_c^\infty(SU_{p,q})$ corresponding to the element $\mathfrak{X}$. Hence, the representation $(\varrho_j, \mathcal{G}_j)$ can be uniquely extended to a representation of $\mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}))$.
2. One may use, instead of the Gårding domain $\mathcal{G}_j$, the space of *well-behaved vectors*, (*analytic vectors*) [23], dense in $\mathcal{H}_j$, which was introduced by Harish-Chandra. An element $\psi \in \mathcal{H}_j$ is said to be well-behaved under ρ_j if the mapping $g \mapsto \rho_j(g)\psi$ is an analytic mapping of $SU_{p,q}$ onto $\mathcal{H}_j$. The use of this space resolves the lack of $SU_{p,q}$-invariance of the Gårding domain under the unique extension of the representation $(\varrho_j, \mathcal{G}_j)$ to a representation of $\mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}))$. Denoting the analytic vector space by $\mathcal{A}_j$, this unique extension to a representation of $\mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}))$ will be denoted by $(\varrho_j, \mathcal{A}_j)$.

10. Invariant eigendistributions on $SU_{p,q}$

Let, as before, $\mathcal{Z}(SU_{p,q})$ denote the center of $SU_{p,q}$ and $\mathfrak{Z}$ denote the center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}))$. The representation $(\rho_j, \mathcal{H}_j)$ is a *quasi-simple* representation of $SU_{p,q}$ if there exist $\eta \in \mathrm{Hom}(\mathcal{Z}(SU_{p,q}), \mathbb{C}-\{0\})$, called the *central character* of $(\rho_j, \mathcal{H}_j)$, and $\chi \in \mathrm{Hom}(\mathfrak{Z}, \mathbb{C})$, called the *infinitesimal character* of $(\rho_j, \mathcal{H}_j)$, such that the following two conditions hold:

$$\begin{aligned} &\rho_j(z)\psi = \eta_{\omega_j}(z)\psi, \quad z \in \mathcal{Z}(SU_{p,q}),\ \psi \in \mathcal{H}_j, \\ &\varrho_j(\mathfrak{z})\overset{\circ}{\psi}_f = \chi_{\omega_j}(\mathfrak{z})\overset{\circ}{\psi}_f, \quad \mathfrak{z} \in \mathfrak{Z},\ \overset{\circ}{\psi}_f \in \mathcal{G}_j, \end{aligned} \tag{10.1}$$

where ω_j is the equivalence class containing the representation $(\rho_j, \mathcal{H}_j)$ of $SU_{p,q}$. For $\psi \in \mathcal{H}_j$, one has $\varrho_j(\mathfrak{z})\psi = \chi_{\omega_j}(\mathfrak{z})\psi$. It is well known [24] that any irreducible unitary representation is quasi-simple.

A linear operator A on the Hilbert space $\mathcal{H}$ is said to be of trace class if for every bounded linear operator B with a bounded linear inverse, $\sum |(\psi_i, B^{-1}AB\psi_i)| < \infty$ for every orthonormal basis $\{\psi_1, \ldots, \psi_n, \ldots\}$. The sum $\sum(\psi_i, B^{-1}AB\psi_i)$ is independent of both $\{\psi_i\}$ and B, and is called the trace of A. A is said to be of the *Hilbert–Schmidt class* if AA^* has a trace, A^* being the *formal adjoint* of A.

Let $(\rho_j, \mathcal{H}_j)$ be an irreducible unitary representation of $SU_{p,q}$ on a Hilbert space $\mathcal{H}_j$. Then for any $f \in C_c^\infty(SU_{p,q})$ satisfying $\int_{SU_{p,q}} |f(g)|^2\, \mathrm{d}\mu(g) < \infty$, where $\mathrm{d}\mu(g)$ is the

Haar measure on $SU_{p,q}$ the bounded linear operator $\rho_{j(f)} = \int_{SU_{p,q}} f(g)\rho_j(g)\,\mathrm{d}\mu(g)$ on $\mathcal{H}_j$ has a trace and is of the Hilbert–Schmidt class [6]. Let $\mathcal{D}$ be the space of all operators of the form $\rho_{j(f)}$. Then, for $g_0 \in SU_{p,q}$,

$$\rho_j(g_0)\rho_{j(f)} = \int_{SU_{p,q}} f(g)\rho_j(g_0 g)\,\mathrm{d}\mu(g) = \int_{SU_{p,q}} f(g_0^{-1}g)\rho_j(g)\,\mathrm{d}\mu(g),$$

and $\rho_{j(f)}\rho_j(g_0^{-1}) = \int_{SU_{p,q}} f(gg_0)\rho_j(g)\,\mathrm{d}\mu(g)$. That is, $\rho_j(g_0)\rho_{j(f)}$ and $\rho_{j(f)}\rho_j(g_0^{-1})$ are in $\mathcal{D}$ for each $g_0 \in SU_{p,q}$. Define the linear functional T_{ω_j} on $C_c^\infty(SU_{p,q})$, called the *global character* (or *distributional character*) of $(\rho_j, \mathcal{H}_j)$, by

$$T_{\omega_j}(f) = \mathrm{tr}(\rho_{j(f)}) = \sum_{i\geq 1}\int_{SU_{p,q}} f(g)(\rho_j(g)(\psi_i), \psi_i)\,\mathrm{d}\mu(g), \quad \rho_j \in \omega_j, \tag{10.2}$$

where $\{\psi_i\}_{i\geq 1}$ is an orthonormal basis of $\mathcal{H}_j$. One can see that this global character T_{ω_j} does not vary within an equivalence class of representations of $SU_{p,q}$. Where $(\rho_j, \mathcal{H}_j)$ and $(\rho'_j, \mathcal{H}'_j)$, are two equivalent representations of $SU_{p,q}$, there exists an isometry A from $\mathcal{H}_j$ onto $\mathcal{H}'_j$ such that $\rho_j(g) = A^{-1}\rho'_j(g)A\ \forall g \in SU_{p,q}$. Thus

$$\rho_{j(f)} = \int_{SU_{p,q}} f(g)\rho_j(g)\,\mathrm{d}\mu(g) = \int_{SU_{p,q}} f(g)A^{-1}\rho'_j(g)A\,\mathrm{d}\mu(g) = A^{-1}\rho'_{j(f)}A.$$

Taking the trace on both sides, one has $T_{\rho_j}(f) = T_{\rho'_j}(f),\ \rho_j, \rho'_j \in \omega_j$. One finds [10] that T_{ω_j} is a distribution in the sense of Laurent Schwartz.

Lemma 10.1. *T_{ω_j} is conjugation-invariant (central) under $SU_{p,q}$.*

Proof. Let $f, f' \in C_c^\infty(SU_{p,q})$ such that, given a $g_0 \in SU_{p,q},\ f'(g) = f(g_0 g g_0^{-1})\ \forall g \in SU_{p,q}$. Then

$$\begin{aligned}\rho_{j(f')} &= \int_{SU_{p,q}} f'(g)\rho_j(g)\,\mathrm{d}\mu(g) = \int_{SU_{p,q}} f(g_0 g g_0^{-1})\rho_j(g)\,\mathrm{d}\mu(g)\\ &= \int_{SU_{p,q}} f(g)\rho_j(g_0^{-1} g g_0)\,\mathrm{d}\mu(g) = \rho_j(g_0^{-1})\rho_{j(f)}\rho_j(g_0).\end{aligned}$$

Taking the trace on both sides, one has $T_{\omega_j}(f') = T_{\omega_j}(f)$. □

Now, by definition, the *formal transpose* $P^{\mathrm{t}}(g, (\partial/\partial g))$ of a differential operator $P(g, (\partial/\partial g))$ on $SU_{p,q}$ satisfies $(P(g, (\partial/\partial g))\psi, \phi) = (\psi, P^{\mathrm{t}}(g, (\partial/\partial g))\phi)$, where $\psi \in C^\infty(SU_{p,q})$ and $\phi \in C_c^\infty(SU_{p,q})$ (or $\psi \in C_c^\infty(SU_{p,q})$ and $\phi \in C^\infty(SU_{p,q})$). That is,

$$\int_{SU_{p,q}} \left(P\left(g, \frac{\partial}{\partial g}\right)\psi\right)(g)\phi(g)\,\mathrm{d}\mu(g) = \int_{SU_{p,q}} \psi(g)\left(P^{\mathrm{t}}\left(g, \frac{\partial}{\partial g}\right)\phi\right)(g)\,\mathrm{d}\mu(g).$$

For example, the transpose of $\partial/\partial g$ is $-\partial/\partial g$. If T_{ω_j} is a distribution of the form $\psi(g)\,\mathrm{d}\mu(g)$ (Radon measure) then one has that $P(g, (\partial/\partial g))T_{\omega_j}$ would be $(P(g, (\partial/\partial g))\psi)(g)\,\mathrm{d}\mu(g)$.

Thus motivated by the invariance property of differential operators (see Section 5) one defines $P(g,(\partial/\partial g))T_{\omega_j}$ to be the distribution given by

$$\left(P\left(g,\frac{\partial}{\partial g}\right)T_{\omega_j}\right)(\phi)=T_{\omega_j}\left(P^{\mathrm{t}}\left(g,\frac{\partial}{\partial g}\right)\phi\right),\quad \phi\in C_{\mathrm{c}}^{\infty}(SU_{p,q}). \tag{10.3}$$

Lemma 10.2. *Let $\tilde{X}$ be a left-invariant vector field on $SU_{p,q}$. Then, as a differential operator $\tilde{X}^t=-\tilde{X}$.*

Proof. Let $\psi\in C^{\infty}(SU_{p,q})$ and $\phi\in C_{\mathrm{c}}^{\infty}(SU_{p,q})$ (or $\psi\in C_{\mathrm{c}}^{\infty}(SU_{p,q})$ and $\phi\in C^{\infty}(SU_{p,q})$). Then

$$\begin{aligned}\int_{SU_{p,q}}(\tilde{X}\psi)(g)\phi(g)\,\mathrm{d}\mu(g)&=\int_{SU_{p,q}}\frac{\mathrm{d}}{\mathrm{d}t}[\psi(g\,\mathrm{e}^{tX})]_{t=0}\phi(g)\,\mathrm{d}\mu(g)\\&=\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{SU_{p,q}}\psi(g\,\mathrm{e}^{tX})\phi(g)\,\mathrm{d}\mu(g)\right]_{t=0}\\&=\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{SU_{p,q}}\psi(g)\phi(g\,\mathrm{e}^{-tX})\,\mathrm{d}\mu(g)\right]_{t=0}\\&=\int_{SU_{p,q}}\psi(g)(-\tilde{X}\phi)(g)\,\mathrm{d}\mu(g).\end{aligned}$$

Hence from the definition, it follows that $\tilde{X}^t=-\tilde{X}$. □

Lemma 10.3. *Let $(\rho_j,\mathcal{H}_j)$ be an irreducible unitary representation of $SU_{p,q}$. Then every matrix coefficient of ρ_j of the form $g\mapsto(\rho_j(g)\psi_1,\psi_2)$ transforms under a left-invariant vector field $\tilde{\mathfrak{X}}$, $\mathfrak{X}\in\mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}))$ as*

$$\tilde{\mathfrak{X}}(\rho_j(g)\psi_1,\psi_2)=(\rho_j(g)\varrho_j(\mathfrak{X})\psi_1,\psi_2),\quad g\in SU_{p,q},\ \psi_1,\psi_2\in\mathcal{H}_j.$$

Proof. Let $\tilde{X}$ be a left-invariant vector field induced by $X\in\mathfrak{su}_{p,q}$. Then by the definition of $\tilde{X}$ one has

$$\begin{aligned}\tilde{X}(\rho_j(g)\psi_1,\psi_2)&=\frac{\mathrm{d}}{\mathrm{d}t}[(\rho_j(g\,\mathrm{e}^{tX})\psi_1,\psi_2)]_{t=0}=\frac{\mathrm{d}}{\mathrm{d}t}[(\rho_j(g)\rho_j(\mathrm{e}^{tX})\psi_1,\psi_2)]_{t=0}\\&=\frac{\mathrm{d}}{\mathrm{d}t}[(\rho_j(\mathrm{e}^{tX})\psi_1,\rho_j^*(g)\psi_2)]_{t=0}=(\varrho_j(X)\psi_1,\rho_j^*(g)\psi_2)\\&=(\rho_j(g)\varrho_j(X)\psi_1,\psi_2).\end{aligned}$$

If one iterates this equality then one has, for any $\mathfrak{X}\in\mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}))$, $\tilde{\mathfrak{X}}(\rho_j(g)\psi_1,\psi_2)=(\rho_j(g)\varrho_j(X)\psi_1,\psi_2)$. □

Proposition 10.4. *Let T_{ω_j} be the global character of an irreducible unitary representation $(\rho_j,\mathcal{H}_j)$ of $SU_{p,q}$. Let $\mathfrak{z}\in\mathfrak{Z}$, the center of $\mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}))$. If $\mathfrak{z}$ is considered as a left-invariant differential operator, then $\mathfrak{z}T_{\omega_j}=\chi_{\omega_j}(\mathfrak{z})T_{\omega_j}$, implying that T_{ω_j} is an invariant eigendistribution for $\mathfrak{Z}$ on $SU_{p,q}$.*

Proof. From (10.3), we have $(\mathfrak{z}T_{\omega_j})(f) = T_{\omega_j}(\mathfrak{z}^t f),\ f \in C_c^\infty(SU_{p,q})$. Let $\{\psi_i\}_{i\geq 1}$ be an orthonormal basis of $\mathcal{H}_j$. Then one has

$$\begin{aligned}
&(\rho_{j(\mathfrak{z}^t f)}\psi_i, \psi_k) \\
&= \left(\int_{SU_{p,q}} (\mathfrak{z}^t f)(g)\rho_j(g)\,\mathrm{d}\mu(g)\psi_i, \psi_k\right) = \int_{SU_{p,q}} ((\mathfrak{z}^t f)(g)\rho_j(g)\psi_i, \psi_k)\,\mathrm{d}\mu(g) \\
&= \int_{SU_{p,q}} (\rho_j(g)\psi_i, \psi_k)(\mathfrak{z}^t f)(g)\,\mathrm{d}\mu(g) = \int_{SU_{p,q}} \mathfrak{z}(\rho_j(g)\psi_i, \psi_k) f(g)\,\mathrm{d}\mu(g) \\
&= \int_{SU_{p,q}} (\rho_j(g)\varrho_j(\mathfrak{z})\psi_i, \psi_k) f(g)\,\mathrm{d}\mu(g) = \int_{SU_{p,q}} (\rho_j(g)\chi_{\omega_j}(\mathfrak{z})\psi_i, \psi_k) f(g)\,\mathrm{d}\mu(g) \\
&= \chi_{\omega_j}(\mathfrak{z}) \int_{SU_{p,q}} f(g)(\rho_j(g)\psi_i, \psi_k)\,\mathrm{d}\mu(g) = \chi_{\omega_j}(\mathfrak{z})(\rho_{j(f)}\psi_i, \psi_k).
\end{aligned}$$

Taking the trace on both sides, one has $T_{\omega_j}(\mathfrak{z}^t f) = \chi_{\omega_j}(\mathfrak{z})T_{\omega_j}(f) = (\mathfrak{z}T_{\omega_j})(f)$. □

Any distribution which satisfies the above proposition is said to be an *eigendistribution* of $\mathfrak{Z}$ on $SU_{p,q}$. Also, a distribution T on $SU_{p,q}$ is said to be $\mathfrak{Z}$-*finite* if the space spanned by $(\mathfrak{z}T_{\omega_j})$ is finite-dimensional. By the above proposition, one has that T_{ω_j} is an eigendistribution of $\mathfrak{Z}$. Because T_{ω_j} is conjugate-invariant under $SU_{p,q}$, T_{ω_j} is an *invariant eigendistribution* of $\mathfrak{Z}$ on $SU_{p,q}$. Also, since the center $\mathfrak{Z}$ is finite-dimensional, T_{ω_j} is $\mathfrak{Z}$-finite.

A complex-valued function on an open subset U of $SU_{p,q}$, is *locally summable* (*locally integrable*) if it is summable on every compact subset of U with respect to the Haar measure of $SU_{p,q}$. According to Harish-Chandra's *regularity theorem* [10], an invariant eigendistribution is represented by a locally $\mathcal{L}^1$ function on a connected reductive group. In terms of $SU_{p,q}$ this theorem may be stated as follows.

Theorem 10.5 (Harish-Chandra [5–14]). *There exists a locally summable function F_{ω_j} on $SU_{p,q}$ which is analytic on $SU'_{p,q}$ such that the invariant eigendistribution T_{ω_j} is given by $T_{\omega_j}(f) = \int_{SU_{p,q}} f(g)F_{\omega_j}(g)\,\mathrm{d}\mu(g)\,\forall f \in C_c^\infty(SU_{p,q})$. The function F_{ω_j} is uniquely determined on $SU'_{p,q}$ by these properties.*

By definition, the global character T_{ω_j} is given by, for all $f \in C_c^\infty(SU_{p,q})$ and for an orthonormal basis $\{\psi_i\}_{i\geq 1}$ of $\mathcal{H}_j$,

$$T_{\omega_j}(f) = \mathrm{tr}(\rho_{j(f)}) = \sum_{i\geq 1}\int_{SU_{p,q}} f(g)(\rho_j(g)\psi_i, \psi_i)\,\mathrm{d}\mu(g).$$

Hence in view of Theorem 10.5, it seems reasonable to take $F_{\omega_j}(g)$ as

$$F_{\omega_j}(g) = \sum_{i\geq 1}(\rho_j(g)\psi_i, \psi_i) \quad \forall g \in SU_{p,q}. \tag{10.4}$$

Proposition 10.6. *The locally summable function F_{ω_j} on $SU_{p,q}$ is an invariant eigendistribution of $\mathfrak{Z}$ on $SU_{p,q}$. That is,* $(\mathfrak{z}F_{\omega_j})(g) = \chi_{\omega_j}(\mathfrak{z})F_{\omega_j}(g)\ \forall g \in SU_{p,q}$.

Proof. The invariance follows from the conjugation invariance of the trace of $(\rho_j(g)\psi_i, \psi_k)$ under $SU_{p,q}$. That it is an eigendistribution can be seen as follows. From Proposition 10.4, $(\mathfrak{z}T_{\omega_j})(f) = \chi_{\omega_j}(\mathfrak{z})T_{\omega_j}(f)\ \forall f \in C_{\mathrm{c}}^{\infty}(SU_{p,q})$. Now, from Theorem 10.5,

$$(\mathfrak{z}T_{\omega_j})(f) = T_{\omega_j}(\mathfrak{z}^t f) = \int_{SU_{p,q}} (\mathfrak{z}^t f)(g)F_{\omega_j}(g)\,\mathrm{d}\mu(g) = \int_{SU_{p,q}} f(g)(\mathfrak{z}F_{\omega_j})(g)\,\mathrm{d}\mu(g),$$

and $\chi_{\omega_j}T_{\omega_j}(f) = \chi_{\omega_j}(\mathfrak{z})\int_{SU_{p,q}} f(g)F_{\omega_j}(g)\,\mathrm{d}\mu(g)$. Hence $(\mathfrak{z}F_{\omega_j})(g) = \chi_{\omega_j}(\mathfrak{z})F_{\omega_j}(g)$. □

The uniqueness of this invariant eigendistribution F_{ω_j} is given by the following theorem [5].

Theorem 10.7 (Harish-Chandra [5–14]). *Given an element $\xi_{(j)\alpha_{u_j}} \in H'^{*}_{(j)}$ which corresponds to a given representation $(\rho_j, \mathcal{H}_j) \in \omega_j$ there exists exactly one invariant eigendistribution F_{ω_j} of $\mathfrak{Z}$ on $SU_{p,q}$. This distribution has the following properties*:

1. *Where the analytic functions $D_{n-1}(g)$ on $SU_{p,q}$ is as defined in Section* 4, $\sup_{g\in SU'_{p,q}} |D_{n-1}(g)|^{1/2}|F_{\omega_j}(g)| < \infty$.
2. *For $h_{(j)} \in H'_{(j)}$*,

$$F_{\omega_j}(h_{(j)}) = (\Delta_{(j)}(h_{(j)}))^{-1} \sum_{w\in W(SU_{p,q},H'_{(j)})} \mathrm{sgn}(w)\xi_{(j)\alpha_{u_j}}(wh_{(j)}).$$

Now, from (7.8) and Proposition 10.6, one has, for $h_{(j)} \in H'_{(j)}$,

$$((\Delta_{(j)}(h_{(j)}))^{-1}\gamma(\mathfrak{z}) \circ \Delta_{(j)}(h_{(j)}))F_{\omega_j}(h_{(j)}) = \chi_{\omega_j}(\mathfrak{z})F_{\omega_j}(h_{(j)}).$$

In other words,

$$\gamma(\mathfrak{z})(\Delta_{(j)}(h_{(j)})F_{\omega_j})(h_{(j)}) = \chi_{\omega_j}(\mathfrak{z})\Delta_{(j)}(h_{(j)})F_{\omega_j}(h_{(j)}),$$

where γ is the Harish-Chandra homomorphism. Let

$$\Xi_j(h_{(j)}) := \epsilon_{(j)\mathrm{R}}(h_{(j)})\Delta_{(j)}(h_{(j)})F_{\omega_j}(h_{(j)}), \tag{10.5}$$

where $\epsilon_{(j)\mathrm{R}}(h_{(j)})$ is as defined in (4.9). Then from Theorem 10.7, for $h_{(j)} \in H'_{(j)}$,

$$\Xi_j(h_{(j)}) = \epsilon_{(j)\mathrm{R}}(h_{(j)}) \sum_{w\in W(SU_{p,q},H'_{(j)})} \mathrm{sgn}(w)\xi_{(j)\alpha_{u_j}}(h_{(j)}). \tag{10.6}$$

Using (4.10), one has $\epsilon_{(j)\mathrm{R}}(wh_{(j)})\Delta_{(j)}(wh_{(j)}) = \epsilon_R(w)\epsilon_{(j)\mathrm{R}}(h_{(j)})\Delta_{(j)}(h_{(j)})$. Then

$$\Xi_j(wh_{(j)}) = \epsilon_R(w)\Xi_j(h_{(j)}),$$

and by (4.10) Ξ_j is skew symmetric under the symmetric groups S_{p-j} and S_{q-j}, symmetric under S_j, and even under $\mathbf{P}(j)$. Furthermore, from (10.5) and Proposition 10.6, one obtains

$$\gamma(\mathfrak{z})\Xi_j(h_{(j)}) = \chi_{\omega_j}(\mathfrak{z})\Xi_j(h_{(j)}), \quad h_{(j)} \in H'_{(j)}. \tag{10.7}$$

Using (8.1) and (10.6), one obtains

$$\prod_{k=1}^{n}\left(h_{(j)k}\frac{\partial}{\partial h_{(j)k}} - u_{(j)k}\right)\Xi_j(h_{(j)}) = 0. \tag{10.8}$$

Solving for $\Xi_j(h_{(j)})$, one obtains a general solution

$$\Xi_j(h_{(j)}) = \sum_{\sigma\in S_n} p_\sigma(h_{(j)1},\dots,h_{(j)n})\prod_{k=1}^{n}(h_{(j)k})^{\sigma(u_{(j)k})},$$

where $p_\sigma(\cdots)$ denotes a polynomial in the respective argument $(\cdots)$.

The invariant eigendistribution F_{ω_j} of $\mathfrak{Z}$ on $SU_{p,q}$ is also an eigendistribution of the center $\mathcal{Z}(SU_{p,q})$ of $SU_{p,q}$ since from (10.1) and (10.4),

$$F_{\omega_j}(gz) = \sum_{i\geq 1}(\rho_j(gz)\psi_i,\psi_i) = \eta_{\omega_j}(z)F_{\omega_j}(g) \quad \forall g\in SU_{p,q},\ z\in\mathcal{Z}(SU_{p,q}).$$

As in Section 2, the center of $SU_{p,q}$ is given by $\mathcal{Z}(SU_{p,q}) = \{\wp_0^k I_n : \wp_0 = \mathrm{e}^{\iota(2\pi/n)k},\ k\in\mathbb{Z}\}$. Since, for $\wp\in\mathcal{Z}(SU_{p,q})$, $\eta_{\omega_j}(\wp) = \wp_0^m$, $m\in\mathbb{Z}$ one has $F_{\omega_j}(g\wp) = \wp_0^m F_{\omega_j}(g)\,\forall g\in SU_{p,q}$. Furthermore, F_{ω_j} can be extended to the eigendistribution F_{ϖ_j} on $U_{p,q}$, where ϖ_j denotes the class of unitary representations of $U_{p,q}$ which contains the unitary representations induced by the unitary representations of $SU_{p,q}$ in the class ω_j. If $\zeta\in\mathbf{T} = \{z\in\mathbb{C} : |z| = 1\}$, then the mapping $(\zeta, g)\mapsto g\zeta$ maps $\mathbf{T}\times SU_{p,q}$ onto $U_{p,q}$. Thus by using (10.1) and (10.4), one obtains for $U_{p,q}$, $g\in SU_{p,q}$, $F_{\varpi_j}(g\zeta) = \zeta^m F_{\omega_j}(g)$. All invariant eigendistributions on $SU_{p,q}$ which are also global characters of a quasi-simple irreducible representation $(\rho_j,\mathcal{H}_j)$ of $SU_{p,q}$ (or $U_{p,q}$) fulfills the above condition for some m. That is, one has the following lemma.

Lemma 10.8. *If* $\wp\in\mathcal{Z}(SU_{p,q})$, $f_\wp\in C_{\mathrm{c}}^\infty(SU_{p,q})$ *such that* $f(g\wp^{-1}) = f_\wp(g)\,\forall g\in SU_{p,q}$, *then* $\wp_0^m T_{\omega_j}(f) = T_{\omega_j}(f_\wp)$.

Proof. We have

$$\begin{aligned}\wp_0^m T_{\omega_j}(f) &= \int_{SU_{p,q}} f(g)\wp_0^m F_{\omega_j}(g)\,\mathrm{d}\mu(g) = \int_{SU_{p,q}} f(g)F_{\omega_j}(g\wp)\,\mathrm{d}\mu(g)\\ &= \int_{SU_{p,q}} f(g\wp^{-1})F_{\omega_j}(g)\,\mathrm{d}\mu(g) = \int_{SU_{p,q}} f_\wp(g)F_{\omega_j}(g)\,\mathrm{d}\mu(g)\\ &= T_{\omega_j}(f_\wp).\end{aligned}$$

□

Remark. However, there may also exist invariant eigendistributions on $SU_{p,q}$, for certain choices of p and q, which do not satisfy the above condition for any m. An example of such a distribution for $SU(1,1)$ can be found as a linear combination of the characters of two non-equivalent irreducible unitary representations with the same infinitesimal character χ.

Proposition 10.9. *For every $f \in C_{\mathrm{c}}^{\infty}(SU_{p,q})$ there exist functions $f_m \in C_{\mathrm{c}}^{\infty}(SU_{p,q})$, $m \in \mathbb{Z}$ such that $f_m(g\wp) = \wp_0^m f(g)$ for all $\wp \in \mathcal{Z}(SU_{p,q})$, $g \in SU_{p,q}$, and such that f can be uniquely expressed as $f = f_0 + \cdots + f_{n-1}$. The mapping $f \mapsto f_m$ is continuous with respect to the usual topology of $C_{\mathrm{c}}^{\infty}(SU_{p,q})$.*

Proof. Let $\alpha_\wp : C_{\mathrm{c}}^{\infty}(SU_{p,q}) \to C_{\mathrm{c}}^{\infty}(SU_{p,q})$ be the mapping defined by $(\alpha_\wp f)(g) = f(g\wp)$, where $\wp \in \mathcal{Z}(SU_{p,q})$, $g \in SU_{p,q}$, $f \in C_{\mathrm{c}}^{\infty}(SU_{p,q})$. Then, since $\wp_0^n = 1$, one has that $\alpha_\wp^n = I = \alpha_{\wp^{-1}}\alpha_\wp = \alpha_\wp\alpha_{\wp^{-1}}$, where I is identity mapping on $C_{\mathrm{c}}^{\infty}(SU_{p,q})$. Hence, since for non-identity $\wp \in \mathcal{Z}(SU_{p,q})$, the set

$$\{\wp^0 = \wp^n, \wp^1, \ldots, \wp^{n-1}\}$$

forms the set of nth roots of identity, one obtains $\alpha_\wp^n - I = \prod_{k=0}^{n-1}(\alpha_\wp - \wp_0^k I) = 0$. After taking the derivative with respect to $\alpha_\wp$ one has

$$\sum_{m=0}^{n-1} \prod_{\substack{k=0 \\ k\neq m}}^{n-1} (\alpha_\wp - \wp_0^k I) = n\alpha_\wp^{n-1} = n\alpha_{\wp^{-1}}.$$

In other words, one finds that $\sum_{m=0}^{n-1} F_m = I$, where $F_m := (1/n)\alpha_\wp \prod_{k=0,\, k\neq m}^{n-1}(\alpha_\wp - \wp_0^k I)$, such that $(\alpha_\wp - \wp_0^m)F_m = 0$. That is, $\alpha_\wp F_m = \wp_0^m F_m$. Let $f_m := F_m f$. Then $\alpha_\wp f_m = \wp_0^m f_m$. Since $F_0 + \cdots + F_{n-1} = I$, one obtains $f_0 + \cdots + f_{n-1} = f$, which is the desired decomposition. The decomposition is unique as it depends only upon $F_0 + \cdots + F_{n-1} = I$. The mapping $f \mapsto f_m = F_m f$ is clearly continuous by the continuity of f on $SU_{p,q}$ and hence the continuity of F_m. Furthermore, $f_m(g\wp) = (\alpha_\wp f_m)(g) = \wp^m f_m(g)$. □

Proposition 10.10. *For every distribution F on $SU_{p,q}$ there exist distributions F_m, $m \in \mathbb{Z}$ on $SU_{p,q}$ such that $F_m(g\wp) = \wp_0^{-m} F_m(g)$ for all $g \in SU_{p,q}$, $\wp \in \mathcal{Z}(SU_{p,q})$, and such that F can be uniquely expressed as $F = F_0 + \cdots + F_{n-1}$. Where F is an invariant eigendistribution, all F_m are also invariant eigendistributions.*

Proof. Let F be a distribution on $SU_{p,q}$. Let F_m, $m = 0, \ldots, n-1$ be distributions on $SU_{p,q}$ satisfying, where $g \in SU_{p,q}$, $f \in C_{\mathrm{c}}^{\infty}(SU_{p,q})$,

$$\int_{SU_{p,q}} f(g)F_m(g)\,\mathrm{d}\mu(g) = \int_{SU_{p,q}} f_m(g)F(g)\,\mathrm{d}\mu(g).$$

Summing over m and using $f_0 + \cdots + f_{n-1} = f$, one obtains the unique decomposition $F_0 + \cdots + F_{n-1} = F$. Furthermore,

$$\begin{aligned}
&\int_{SU_{p,q}} f(g) F_m(g)\,\mathrm{d}\mu(g) \\
&\quad = \int_{SU_{p,q}} f_m(g) F(g)\,\mathrm{d}\mu(g) \\
&\quad = \int_{SU_{p,q}} f_m(g\wp) F(g\wp)\,\mathrm{d}\mu(g) = \wp_0^m \int_{SU_{p,q}} f_m(g) F(g\wp)\,\mathrm{d}\mu(g) \\
&\quad = \wp_0^m \int_{SU_{p,q}} (\alpha_{\wp^{-1}} f)(g\wp) F(g\wp)\,\mathrm{d}\mu(g) \\
&\quad = \wp_0^m \int_{SU_{p,q}} (\alpha_{\wp^{-1}} f)_{g\wp} F(g\wp)\,\mathrm{d}\mu(g) = \wp_0^m \int_{SU_{p,q}} (\alpha_{\wp^{-1}} f)(g\wp) F_m(g\wp)\,\mathrm{d}\mu(g) \\
&\quad = \wp_0^m \int_{SU_{p,q}} f(g) F_m(g\wp)\,\mathrm{d}\mu(g).
\end{aligned}$$

Hence $\int_{SU_{p,q}} f(g)(F_m(g) - \wp_0^m F_m(g\wp))\,\mathrm{d}\mu(g) = 0$. Therefore, $F_m(g\wp) = \wp^{-m} F_m(g)$. That all F_m are invariant eigendistributions whenever F is an invariant eigendistribution follows immediately from the unique decomposition of F. □

Proposition 10.11. *Let $f, h \in C_c^\infty(SU_{p,q})$. Then there exist $\tilde{f}, \tilde{h} \in C_c^\infty(U_{p,q})$ such that*

$$\int_{SU_{p,q}} f(g)\overline{h(g)}\,\mathrm{d}\mu(g) = \int_{U_{p,q}} \tilde{f}(g)\overline{\tilde{h}(g)}\,\mathrm{d}\mu(g),$$

where $\mathrm{d}\mu$ is the Haar measure on the respective group.

Proof. As in Proposition 10.9, $f = f_0 + \cdots + f_{n-1} \in C_c^\infty(SU_{p,q})$ and $h = h_0 + \cdots + h_{n-1} \in C_c^\infty(SU_{p,q})$ such that for $k \neq l$, $\int_{SU_{p,q}} f_k(x)\overline{h_l(x)}\,\mathrm{d}\mu(x) = 0$. Extend every function f_k and h_k to functions $\tilde{f}_k$ and $\tilde{h}_k$ in $C_c^\infty(U_{p,q})$ such that for $\zeta \in \mathbf{T}$, $g \in SU_{p,q}$, and $0 \le k \le n-1$, $\tilde{f}_k(g\zeta) = \zeta^k f_k(g)$ and $\tilde{h}_k(g\zeta) = \zeta^k h_k(g)$. Consequently, one has for $k \neq l$, $\int_{U_{p,q}} \tilde{f}_k(g)\overline{\tilde{h}_l(g)}\,\mathrm{d}\mu(g) = 0$. Define $\tilde{f} = \tilde{f}_0 + \cdots + \tilde{f}_{n-1}$ and $\tilde{h} = \tilde{h}_0 + \cdots + \tilde{h}_{n-1}$. Then, for any $f, h \in C_c^\infty(SU_{p,q})$,

$$\begin{aligned}
\int_{SU_{p,q}} f(g)\overline{h(g)}\,\mathrm{d}\mu(g) &= \sum_{k=0}^{n-1} \int_{SU_{p,q}} f_k(g)\overline{h_k(g)}\,\mathrm{d}\mu(g) = \sum_{k=0}^{n-1} \int_{SU_{p,q}} \tilde{f}_k(g\zeta)\overline{\tilde{h}_k(g\zeta)}\,\mathrm{d}\mu(g) \\
&= \sum_{k=0}^{n-1} \int_{U_{p,q}} \tilde{f}_k(g)\overline{\tilde{h}_k(g)}\,\mathrm{d}\mu(g) = \int_{U_{p,q}} \tilde{f}(g)\overline{\tilde{h}(g)}\,\mathrm{d}\mu(g).
\end{aligned}$$ □

Remark. Let $\tilde{H}_j$, $0 \le j \le p$ denote the Cartan subgroups of $U_{p,q}$. One has then that $H_j = \tilde{H}_j \cap SU_{p,q}$. Similarly, the Cartan subalgebras $\tilde{\mathfrak{h}}_j$ of $U_{p,q}$ are such that $\mathfrak{h}_j = \tilde{\mathfrak{h}}_j \cap \mathfrak{su}_{p,q}$.

One has following correspondences:

$SU_{p,q}$	$U_{p,q}$
$(h_{(j)1},\dots,h_{(j)n}),\qquad h_{(j)1}\cdots h_{(j)n}=1$	$(h_{(j)1},\dots,h_{(j)n})$
$\left(\dfrac{\partial}{\partial y_{(j)1}},\dots,\dfrac{\partial}{\partial y_{(j)n}}\right)$,	$\left(\dfrac{\partial}{\partial x_{(j)1}},\dots,\dfrac{\partial}{\partial x_{(j)n}}\right)$,
$\dfrac{\partial}{\partial y_{(j)k}}:=\dfrac{\partial}{\partial x_{(j)k}}-\dfrac{1}{n}\left(\dfrac{\partial}{\partial x_{(j)1}}+\cdots+\dfrac{\partial}{\partial x_{(j)n}}\right)$,	$x_{(j)k}:=\ln(h_{(j)k}),\quad 1\le k\le n$
$\dfrac{\partial}{\partial y_{(j)1}}+\cdots+\frac{\partial}{\partial y_{(j)n}}=0,\quad 1\le k\le n$	
$(u'_{(j)1},\dots,u'_{(j)n})$,	$(u_{(j)1},\dots,u_{(j)n})$
$u'_{(j)k}:=u_{(j)k}-\dfrac{1}{n}(u_{(j)1}+\cdots+u_{(j)n})$,	
$u'_{(j)1}+\cdots+u'_{(j)n}=0,\quad 1\le k\le n$	

For $j\neq 0$, the parameters that describe the representations in ω_j can be given as

$$(m_{(j)1},\dots,m_{(j)p-j},\sigma_{(j)p-j+1},\dots,\sigma_{(j)p+j},m_{(j)p+j+1},\dots,m_{(j)n}),$$

where $\sigma_{(j)k}$ is given in terms of $r_{(j)k}$ as in (8.2), $\sigma_{(j)p+i}=\bar\sigma_{(j)p-i+1},\ i=1,\dots,j$, and

$$m_{(j)1}\geq\cdots\geq m_{(j)p-j}\geq m_{(j)p+j+1}\geq\cdots\geq m_{(j)n}\quad\text{with}\quad m_{(j)k}\in\mathbb{Z}.$$

Let $m_{(j)}:=m_{(j)1}+\cdots+m_{(j)p}+m_{(j)p+j+1}+\cdots+m_{(j)n}$. Then by extending the relation $F_{\omega_j}(gz)=\eta_{\omega_j}(z)F_{\omega_j}(g),\ z\in\mathcal{Z}(SU_{p,q})$ for $SU_{p,q}$ to $U_{p,q}$, one has $F_{\varpi_j}(g\zeta)=\zeta^{m_{(j)}}F_{\varpi_j}(g),\ \zeta\in\mathcal{Z}(U_{p,q}),\ g\in U_{p,q}$. Now, for any fixed $\epsilon\in\mathbb{Z}$, let $m'_{(j)k}:=m_{(j)k}+\epsilon$ for $1\le k\le p$ and $p+j+1\le k\le n$. Let ω'_j (and hence ϖ'_j) contain the representations described by the parameters $m'_{(j)k}$. Then one has from the definition of F_{ϖ_j}, $F_{\varpi'_j}(g)=(\det(g))^{\epsilon}F_{\varpi_j}(g),\ g\in U_{p,q}$. Hence for $g\in SU_{p,q}$ and for any $\epsilon\in\mathbb{Z}$, $F_{\omega'_j}(g)=F_{\omega_j}(g)$. This means that the representations $(\rho'_j,\mathcal{H}'_j)\in\omega'_j$ and $(\rho_j,\mathcal{H}_j)\in\omega_j$ are unitarily equivalent, i.e., one of the $m_{(j)k}$ and one of the $m'_{(j)k}$ can be made zero.

The invariant eigendistribution F_{ω_j} as given in (10.4) is a locally summable function and is analytic on $H'_{(j)},\ j=0,\dots,p$. One can extend its domain of definition to $H'_{(0)}\cup\cdots\cup H'_{(p)}$. The necessary and sufficient conditions that the extended F_{ω_j} is a locally summable function and analytic on $H'_{(0)}\cup\cdots\cup H'_{(p)}$ are given by the differentiability and continuity of $F_{\omega_j}(h_{(j)})$. These conditions imply by (10.5) the following theorem on Ξ_j.

Theorem 10.12. *Let the coordinates ϕ_i,ψ_j,z_k, and $\bar z_m$ be defined as in* (3.8).

Then

$$\left(\lim_{(\phi^-_{(j)},\psi^-_{(j)})} - \lim_{(\phi^+_{(j)},\psi^+_{(j)})}\right)\left(\frac{1}{\iota}\frac{\partial}{\partial\phi_{(j)p-j}}\right)^r \Xi_j(h_{(j)})$$
$$= \left(\lim_{(\phi^-_{(j)},\psi^-_{(j)})} - \lim_{(\phi^+_{(j)},\psi^+_{(j)})}\right)\left(\frac{1}{\iota}\frac{\partial}{\partial\psi_{(j)q-j}}\right)^r \Xi_j(h_{(j)}) = 0,$$
$$\left[\lim_{(\phi^\pm_{(j)},\psi^\pm_{(j)})}\left(\frac{1}{\iota}\frac{\partial}{\partial\phi_{(j)p-j}}\right)^r - \lim_{\psi^\pm_{(j)}}\left(\frac{1}{\iota}\frac{\partial}{\partial\psi_{(j)q-j}}\right)^r\right]\Xi_j(h_{(j)})$$
$$= \lim_{t_{(j+1)j+1}\to 0}\left[\left(\frac{\partial}{\partial z_{(j+1)j+1}}\right)^r - \left(\frac{\partial}{\partial \bar{z}_{(j+1)j+1}}\right)^r\right]\Xi_{j+1}(h_{(j+1)}),$$

where $\phi^\pm_{(j)} := \phi_{(j)p-j} \to \theta_{(j+1)j+1} \pm 0$ *and* $\psi^\pm_{(j)} := \psi_{(j)q-j} \to \theta_{(j+1)j+1} \pm 0$.

Proof. Follows by induction on r. □

11. Explicit construction of invariant eigendistributions

The function Ξ_{ω_j} introduced in Section 10 satisfies

$$\Xi_{\omega_j}(wh_{(j)}) = \epsilon_R(w)\Xi_j(h_{(j)}), \quad h_{(j)} \in H'_{(j)},\ w \in W(SU_{p,q}, H'_{(j)}),$$

and is skew symmetric under the symmetry groups S_{p-j} and S_{q-j}, symmetric under the symmetry group S_j, and *even* under $\mathbf{P}_j$. Using these properties, one can construct Ξ_j, hence F_{ω_j}, explicitly. Define the sets

$$I_n = \{1, 2, \dots, n\}, \qquad I_p = \{i_1, i_2, \dots, i_p\}, \quad I_q = \{i_{p+1}, i_{p+2}, \dots, i_n\} = I_n - I_p,$$

where $i_k \in I_n$ are assumed to satisfy $i_i < i_2 < \cdots < i_p < i_{p+1} < i_{p+2} < \cdots < i_n$. For $j = 1, 2, \dots, p$, define j-tuples from the elements of I_p and I_q, respectively, as $(i_a, i_{a+1}, \dots, i_{a+j-1})$, $a = 1, 2, \dots, p-j+1$ and $(i_b, i_{b+1}, \dots, i_{b+j-1})$, $b = p+1, p+2, \dots, n-j+1$. Let

$$[\sigma_p] = (-1)^{pq}\mathrm{sgn}\begin{pmatrix} 1 & 2 & \cdots & p & p+1 & p+2 & \cdots & n \\ i_1 & i_2 & \cdots & i_p & i_{p+1} & i_{p+2} & \cdots & i_n \end{pmatrix},$$

$$[{}_j\sigma_a] = \mathrm{sgn}\begin{pmatrix} i_1 & i_2 & \cdots & i_{p-j} & i_{p-j+1} & \cdots & i_p \\ i'_1 & i'_2 & \cdots & i'_{p-j} & i_a & \cdots & i_{a+j-1} \end{pmatrix},$$

$$[{}_j\sigma_b] = \mathrm{sgn}\begin{pmatrix} i_{p+1} & i_{p+2} & \cdots & i_{p+j} & i_{p+j+1} & \cdots & i_n \\ i_b & i_{b+1} & \cdots & i_{b+j-1} & i'_{p+j+1} & \cdots & i'_n \end{pmatrix},$$

where for $1 \le k \le p$, $i'_k \in I_p - \{i_a, i_{a+1}, \dots, i_{a+j-1}\} := I_a$ such that $i'_1 < i'_2 < \cdots < i'_{p-j}$, and, for $p+1 \le k \le n$, $i'_k \in I_q - \{i_b, i_{b+1}, \dots, i_{b+j-1}\} := I_b$ such that $i'_{p+j+1} < i'_{p+j+2} < \cdots < i'_n$.

As noted earlier, the Cartan subgroup H_j induces discrete series representations for $j = 0$ and continuous series representations for $j = 1, 2, \ldots, p$. However, for $j \neq 0$, one can also obtain a discrete representation from the continuous representation as the complex parameters in (8.2) are chosen to be integers, i.e., as $\sigma_{(j)k} \in \mathbb{C}$ are chosen to be $m_{(j)k} \in \mathbb{Z}$ for $p-j+1 \leq k \leq p+j$. Similarly, a continuous representation corresponding to a Cartan subgroup $H_m,\ m \neq 0$ can be obtained from that induced by $H_k,\ m < k \neq 0$. Thus in order to include such representations *in the limit*, it is necessary to introduce a parameter ϱ such that $0 \leq \varrho \leq j \leq p$. Correspondingly, we say that the representation is of *type*-(ϱ, j) when the replacement of $(j-\varrho)$ complex parameters with integer parameters has been made. Clearly, the representation of type-(ϱ, j) is equivalent to the representation of type-$(\varrho, j'),\ j \neq j'$. This implies that every non-equivalent representation can be identified by ϱ. Consequently, a class of equivalent representations can be denoted by ω_ϱ, and a representation in this class is parameterized by

$$\begin{aligned} \Lambda_\varrho &= \{m_1, \ldots, m_{p-\varrho}, \sigma_{p-\varrho+1}, \ldots, \sigma_p, \sigma_{p+1}, \ldots, m_{p+\varrho+1}, \ldots, m_{n-1}\}, \\ & m_k \in \mathbb{Z},\ \sigma_k \in \mathbb{C} \end{aligned} \tag{11.1}$$

such that $\sigma_{p+i} = \bar{\sigma}_{p-i+1},\ i = 1, \ldots, j$, the bar denoting the complex conjugation.

From (8.3) and (10.6), the function which is skew symmetric with respect to $S_{p-j} \subset W(SU_{p,q}, H'_{(j)})$ can be obtained as

$$\begin{aligned} {}_{I_a}A(\mathrm{e}^{\iota\phi}) &= \sum_{\sigma \in S_{p-j}} [\sigma] \prod_{k=1}^{p-j} (\mathrm{e}^{\iota\phi_{(j)k}})^{m_{i'_{\sigma(k)}}} \\ &= \det \begin{pmatrix} (\mathrm{e}^{\iota\phi_{(j)1}})^{m_{i'_1}} & \cdots & (\mathrm{e}^{\iota\phi_{(j)1}})^{m_{i'_{p-j}}} \\ \vdots & \ddots & \vdots \\ (\mathrm{e}^{\iota\phi_{(j)p-j}})^{m_{i'_1}} & \cdots & (\mathrm{e}^{\iota\phi_{(j)p-j}})^{m_{i'_{p-j}}} \end{pmatrix}, \quad i'_k \in I_a, \end{aligned} \tag{11.2}$$

where $\mathrm{e}^{\iota\phi} := \{\mathrm{e}^{\iota\phi_1}, \ldots, \mathrm{e}^{\iota\phi_{p-j}}\}$, and $[\sigma]$ is the sign of a permutation of $p - j$ elements $\{1, 2, \ldots, p - j\}$. Similarly, the function which is skew symmetric with respect to S_{q-j} is given by

$$\begin{aligned} A_{I_b}(\mathrm{e}^{\iota\psi}) &= \sum_{\sigma \in S_{q-j}} [\sigma] \prod_{k=1}^{q-j} (\mathrm{e}^{\iota\psi_{(j)k}})^{m_{i'_{p+j+\sigma'(k)}}} \\ &= \det \begin{pmatrix} (\mathrm{e}^{\iota\psi_{(j)q-j}})^{m_{i'_{p+j+1}}} & \cdots & (\mathrm{e}^{\iota\psi_{(j)q-j}})^{m_{i'_n}} \\ \vdots & \ddots & \vdots \\ (\mathrm{e}^{\iota\psi_{(j)1}})^{m_{i'_{p+j+1}}} & \cdots & (\mathrm{e}^{\iota\psi_{(j)1}})^{m_{i'_n}} \end{pmatrix}, \quad i'_k \in I_b, \end{aligned} \tag{11.3}$$

where $\mathrm{e}^{\iota\psi} := \{\mathrm{e}^{\iota\psi_{q-j}}, \ldots, \mathrm{e}^{\iota\psi_1}\}$, and $[\sigma]$ is the sign of a permutation of $q - j$ elements $\{1, 2, \ldots, q - j\}$.

Now, one can obtain a function which is symmetric with respect to S_j and even with respect to $\mathbf{P}(j)$ as

$$
{}_{I_a}S^{\varrho}_{I_b}(\mathrm{e}^z) = \sum_{\substack{\sigma\in S_j\\ \sigma(1)<\cdots<\sigma(\varrho)\\ \sigma(\varrho+1)<\cdots<\sigma(j)}} \prod_{k=1}^{\varrho}\left(\frac{\mathrm{e}^{z_{(j)k}}}{|\mathrm{e}^{z_{(j)k}}|}\right)^{\lambda_{i_{a+\sigma(k)-1}}} |\mathrm{e}^{z_{(j)k}}|^{\iota r_{i_{a+\sigma(k)-1}}} \prod_{l=\varrho+1}^{j} \mathrm{sgn}(i_a-i_b)
\times \begin{cases} (\mathrm{e}^{z_{(j)l}\lambda_{i_{a+\sigma(l)-1}}}\,\mathrm{e}^{-\bar{z}_{(j)l}\lambda_{i_{b+\sigma(l)-1}}}), & \epsilon>0\\ (\mathrm{e}^{-\bar{z}_{(j)l}\lambda_{i_{a+\sigma(l)-1}}}\,\mathrm{e}^{z_{(j)l}\lambda_{i_{b+\sigma(l)-1}}}), & \epsilon<0, \end{cases} \tag{11.4}
$$

where $\mathrm{e}^z := \{\mathrm{e}^{z_j}, \mathrm{e}^{z_{j-1}}, \ldots, \mathrm{e}^{z_1}, \mathrm{e}^{-\bar{z}_1}, \mathrm{e}^{-\bar{z}_2}, \ldots, \mathrm{e}^{-\bar{z}_j}\}$, and $\epsilon = \mathrm{sgn}(i_a - i_b)\mathrm{sgn}(t_{(j)k})$, $\varrho + 1 \le k \le j$, $t_{(j)k} = \Re(z_{(j)k})$.

The function Ξ_j (now denoted by $\Xi_{j\Lambda_\varrho}$) for a given I_p is given by

$$
\Xi_{j\Lambda_\varrho}(\mathrm{e}^{\iota\phi}, \mathrm{e}^z, \mathrm{e}^{\iota\psi}) = \sum_{I_a, I_b} [\sigma_a][\sigma_b]_{I_a} A(\mathrm{e}^{\iota\phi})_{I_a} S^{\varrho}_{I_b}(\mathrm{e}^z) A_{I_b}(\mathrm{e}^{\iota\psi}), \tag{11.5}
$$

where Λ_ϱ is as in (11.1). We obtain the following theorem.

Theorem 11.1. *The invariant eigendistribution F_{ω_j} on $SU_{p,q}$ (now denoted by $F_{j\Lambda_\varrho}$) as given in Theorem* 10.7 *can be now given as*

Discrete series :

$$
\begin{aligned}
F_{j\Lambda_0}(\mathrm{e}^{\iota\phi}, \mathrm{e}^z, \mathrm{e}^{\iota\psi}) &= \left(\frac{\epsilon_{(j)\mathrm{R}}(h_{(j)})}{\Delta_{(j)}(h_{(j)})}\right) \Xi_{j\bar{\Lambda}_0}(\mathrm{e}^{\iota\phi}, \mathrm{e}^z, \mathrm{e}^{\iota\psi})\\
&= \left(\frac{\epsilon_{(j)\mathrm{R}}(h_{(j)})}{\Delta_{(j)}(h_{(j)})}\right) \sum_{I_a, I_b} [{}_j\sigma_a][{}_j\sigma_b]_{I_a} A(\mathrm{e}^{\iota\phi})_{I_a} S^{0}_{I_b}(\mathrm{e}^z) A_{I_b}(\mathrm{e}^{\iota\psi}),\\
0 \le j \le p. &
\end{aligned} \tag{11.6}
$$

Continuous series :

$$
\begin{aligned}
F_{j\Lambda_\varrho}(\mathrm{e}^{\iota\phi}, \mathrm{e}^z, \mathrm{e}^{\iota\psi}) &= \left(\frac{\epsilon_{(j)\mathrm{R}}(h_{(j)})}{\Delta_{(j)}(h_{(j)})}\right) \Xi_{j\bar{\Lambda}_\varrho}(\mathrm{e}^{\iota\phi}, \mathrm{e}^z, \mathrm{e}^{\iota\psi})\\
= \left(\frac{\epsilon_{(j)\mathrm{R}}(h_{(j)})}{\Delta_{(j)}(h_{(j)})}\right) &\sum_{I_a, I_b} [{}_j\sigma_a][{}_j\sigma_b]_{I_a} A(\mathrm{e}^{\iota\phi})_{I_a} S^{\varrho}_{I_b}(\mathrm{e}^z) A_{I_b}(\mathrm{e}^{\iota\psi}), \quad 1 \le \varrho \le j \le p,
\end{aligned} \tag{11.7}
$$

where for $0 \le \varrho \le j \le p$, $\bar{\Lambda}_\varrho = \Lambda_\varrho$ *with* $m_k = l_k - (n-k)$, $k = 1, \ldots, n$, $l_1 > \cdots > l_n$.

Thus the leading term of $\Xi_{j\bar{\Lambda}_\varrho}$ is $\prod_k (h_{(j)k})^{l_k}$ which, when divided by $\Delta_{(j)}(h_{(j)})$, gives one of the desired leading term $\prod_k (h_{(j)k})^{m_k}$ for the character $F_{j\Lambda_\varrho}$ in terms of the highest weights Λ_ϱ, given by (11.1). The expression for the discrete series with $j = 0$ has also been obtained in [4,19–21].

Example. When $p = q = 1$, i.e., for the group $SU_{1,1} \cong SL_2(\mathbb{R}) \cong SO_{2,1}$, we have

$0 \leq \varrho \leq j \leq 1$. The Cartan subgroups are given by

$$H_0 = \left\{ \begin{pmatrix} e^{\iota\phi_1} & 0 \\ 0 & e^{-\iota\phi_1} \end{pmatrix}, \quad \phi_1 \in \mathbb{R} \right\},$$

and

$$H_1 = \left\{ \begin{pmatrix} e^{t_1} & 0 \\ 0 & e^{-t_1} \end{pmatrix}, \quad t_1 \in \mathbb{R} \right\}.$$

All the three cases ($j = \varrho = 0$; $j = 1, \varrho = 0$; $j = \varrho = 1$) can be determined from (11.6) and (11.7), and we get

$$F_{0m_1}(e^{\iota\phi_1}) = \frac{e^{\iota m_1\phi_1}}{e^{\iota\phi_1} - e^{-\iota\phi_1}}, \qquad F_{1m_1}(e^{t_1}) = \frac{e^{-|m_1 t_1|}}{|e^{t_1} - e^{-t_1}|}\mathrm{sgn}(t_1),$$
$$F_{1\lambda_1}(e^{\iota t_1}) = \frac{e^{\iota\lambda_1 t_1} + e^{-\iota\lambda_1 t_1}}{|e^{t_1} - e^{-t_1}|}.$$

These special cases closely agree with the results [14] obtained by direct computation for $SU_{1,1}$. For the group $SU_{2,2}$, there are six character functions given by $F_{0\Lambda_0}$, $F_{1\Lambda_0}$, $F_{2\Lambda_0}$, $F_{1\Lambda_1}$, $F_{2\Lambda_1}$, $F_{2\Lambda_2}$, where $\Lambda_0 = \{m_1, m_2, m_3\}$, $\Lambda_1 = \{m_1, \sigma_2, m_3\}$, $\Lambda_2 = \{\sigma_1, \sigma_2, \sigma_3\}$, $\sigma_k = (\lambda_k, r_k)$. These functions can be directly determined from (11.6) and (11.7).

Theorem 11.2. *The eigendistributions $F_{j\Lambda_\varrho}$ are locally summable functions and are analytic on $H'_0 \cup \cdots \cup H'_p$.*

Proof. One must show that the functions $\Xi_{j\Lambda_\varrho}$ satisfy the conditions given in Theorem 10.12. From Eqs. (11.2)–(11.5), we get

$$\left(\frac{1}{\iota}\frac{\partial}{\partial\phi_{(j)p-j}}\right)^r_{I_a} A(e^{\iota\phi_{(j)}})$$
$$= \left(\frac{1}{\iota}\frac{\partial}{\partial\phi_{(j)p-j}}\right)^r \det\begin{pmatrix} (e^{\iota\phi_{(j)1}})^{m_{(j)i'_1}} & \cdots & (e^{\iota\phi_{(j)1}})^{m_{(j)i'_{p-j}}} \\ \vdots & \ddots & \vdots \\ (e^{\iota\phi_{(j)p-j}})^{m_{(j)i'_1}} & \cdots & (e^{\iota\phi_{(j)p-j}})^{m_{(j)i'_{p-j}}} \end{pmatrix}$$
$$= \sum_{\alpha\in I_a}[\sigma_\alpha](m^r_{(j)\alpha}\, e^{\iota\theta_{(j+1)j+1} m_{(j)\alpha}})\det\begin{pmatrix} e^{\iota\phi_{(j)1} m_{(j)i''_1}} & \cdots & e^{\iota\phi_{(j)1} m_{(j)i''_{p-j-1}}} \\ \vdots & \ddots & \vdots \\ e^{\iota\phi_{(j)p-j-1} m_{(j)i''_1}} & \cdots & e^{\iota\phi_{(j)p-j-1} m_{(j)i''_{p-j-1}}} \end{pmatrix},$$

$i'_k \in I_a$,

$$\left(\frac{1}{\iota}\frac{\partial}{\partial\psi_{(j)q-j}}\right)^r A_{I_b}(\mathrm{e}^{\iota\psi_{(j)}})$$
$$=\left(\frac{1}{\iota}\frac{\partial}{\partial\psi_{(j)q-j}}\right)^r \det\begin{pmatrix} (\mathrm{e}^{\iota\psi_{(j)q-j}})^{m_{(j)k'_2}} & \cdots & (\mathrm{e}^{\iota\psi_{(j)q-j}})^{m_{(j)k'_{q-j}}} \\ \vdots & \ddots & \vdots \\ (\mathrm{e}^{\iota\psi_{(j)2}})^{m_{(j)k'_2}} & \cdots & (\mathrm{e}^{\iota\psi_{(j)2}})^{m_{(j)k'_{q-j}}} \end{pmatrix}$$
$$=\sum_{\beta\in I_b}[\sigma_\beta](m^r_{(j)\beta}\,\mathrm{e}^{\iota\theta_{(j+1)j+1}m_{(j)\beta}})\det\begin{pmatrix} \mathrm{e}^{\iota\psi_{(j)q-j-1}m_{(j)k''_2}} & \cdots & \mathrm{e}^{\iota\psi_{(j)q-j-1}m_{(j)k''_{q-j-1}}} \\ \vdots & \ddots & \vdots \\ \mathrm{e}^{\iota\psi_{(j)2}m_{(j)k''_2}} & \cdots & \mathrm{e}^{\iota\psi_{(j)2}m_{(j)k''_{q-j-1}}} \end{pmatrix},$$
$$k'_s \in I_b,$$

$$\left(\frac{\partial^r}{\partial z^r_{(j+1)j+1}}-\frac{\partial^r}{\partial\bar{z}^r_{(j+1)j+1}}\right)_{I_a} A^{\varrho}_{I_b}(\mathrm{e}^z)=(m^r_{(j)\alpha}-m^r_{(j)\beta})\,\mathrm{e}^{\iota\theta_{(j+1)j+1}(m_{(j)\alpha}+m_{(j)\beta})},$$
$$\alpha\in I_a,\ \beta\in I_b,$$

where
$$[\sigma_\alpha]=\operatorname{sgn}\begin{pmatrix} i'_1 & \cdots & i'_{p-j-1} & i'_{p-j} \\ i''_1 & \cdots & i''_{p-j-1} & \alpha \end{pmatrix},$$
$$[\sigma_\beta]=\operatorname{sgn}\begin{pmatrix} k'_2 & \cdots & k'_{q-j-1} & k'_{q-j} \\ k''_2 & \cdots & k''_{q-j-1} & \beta \end{pmatrix},$$
satisfying $[\sigma_a]\cdot[\sigma_\alpha]\times[\sigma_b]\cdot[\sigma_\beta]=[\sigma_{a,\alpha}]\times[\sigma_{b,\beta}]$, where
$$[\sigma_{a,\alpha}]=\operatorname{sgn}\begin{pmatrix} i_1 & \cdots & i_{a-2} & i_{a-1} & i_a & i_{a+1} & \cdots & i_{a+j-2} & i_{a+j-1} & i_{a+j} & \cdots & i_p \\ i''_1 & \cdots & i''_{a-2} & i'_{i_a} & i'_{i_{a+1}} & i'_{i_{a+2}} & \cdots & i'_{i_{a+j-1}} & \alpha & i''_{a+j} & \cdots & i''_p \end{pmatrix},$$
$$[\sigma_{b,\beta}]=\operatorname{sgn}\begin{pmatrix} i_{p+1} & \cdots & i_{b-2} & i_{b-1} & i_b & i_{b+1} & \cdots & i_{b+j-2} & i_{b+j-1} & i_{b+j} & \cdots & i_{n-1} \\ i''_{p+1} & \cdots & i''_{b-2} & k'_{i_b} & k'_{i_{b+1}} & k'_{i_{b+2}} & \cdots & k'_{i_{b+j-1}} & \beta & i''_{b+j} & \cdots & i''_{n-1} \end{pmatrix}.$$

Hence one can see that the functions $\Xi_{j\Lambda_\varrho}(\mathrm{e}^{\iota\phi},\mathrm{e}^z,\mathrm{e}^{\iota\psi})$ thus obtained indeed satisfy the conditions given in Theorem 10.12, and that the distributions $T_{\omega^\varrho_j}$ on $SU_{p,q}$ defined in Theorem 10.5 are invariant eigendistributions satisfying Proposition 10.6. □

12. Tempered invariant eigendistributions

We now show that the invariant eigendistributions $T_{\omega^0_j}$ and $T_{\omega^\varrho_j}$ are *tempered* [1,11–13]. Let, as before, K be a maximal compact subgroup of $SU_{p,q}$ and $\mathfrak{a}_\mathfrak{p}$ (see Section 3) be a fixed maximal abelian subspace of $\mathfrak{p}$. Define a norm on $\mathfrak{su}_{p,q}$ by putting $\|X\|^2 = -B(X, \theta X)$, $X \in \mathfrak{su}_{p,q}$ (see Section 2). Then since $SU_{p,q} = KA_{\mathfrak{p}}K$, $A_{\mathfrak{p}} = \mathrm{e}^{\mathfrak{a}_{\mathfrak{p}}}$, there exists a unique function σ on $SU_{p,q}$ such that $\sigma(k_1 g k_2) = \sigma(g)$, $k_1, k_2 \in K$, $g \in SU_{p,q}$ and $\sigma(\mathrm{e}^{X_{\mathfrak{p}}}) = \|X_{\mathfrak{p}}\|$, $X_{\mathfrak{p}} \in \mathfrak{a}_{\mathfrak{p}}$, choose $k \in K$ and $a \in A_{\mathfrak{p}}$ such that $aka = k$, and for $g_1, g_2 \in SU_{p,q}$, one obtains $\sigma(a^{-1}) = \sigma(k^{-1}ak) = \sigma(a)$ and $\sigma(g_1 g_2) \leq \sigma(g_1) + \sigma(g_2)$.

Now, for $g \in SU_{p,q}$, define

$$\Phi(g) = \int_K \mathrm{e}^{-\rho(X_{\mathfrak{p}}(gk))}\,\mathrm{d}\mu(k),$$

where $\rho = \frac{1}{2}\sum_{\alpha \in R^+} m_\alpha \alpha$ (see Section 3), m_α being the multiplicity of the weight α, $\mathrm{e}^{X_{\mathfrak{p}}} \in A_{\mathfrak{p}}$. Φ is nothing more than the zonal spherical function on $SU_{p,q}$ corresponding to the trivial linear function on $\mathfrak{a}_{\mathfrak{p}}$. Since any two maximal abelian subspaces of $\mathfrak{p}$ are K-conjugate, Φ is actually independent of the choice of $\mathfrak{a}_{\mathfrak{p}}$. It is well known that the function Φ satisfies the following properties:

1. $\Phi(e) = e$, $\Phi(k_1 g k_2) = \Phi(g) = \Phi(g^{-1})$, $k_1, k_2 \in K$, $g, e \in SU_{p,q}$, e being the identity element.
2. $\Phi(g_1)\Phi(g_2) = \int_K \Phi(g_1 k g_2)\,\mathrm{d}\mu(k)$, $g_1, g_2 \in SU_{p,q}$.
3. $\Phi(a^{-1}) = \Phi(k^{-1}ak) = \Phi(a)$, $a \in A_{\mathfrak{p}}$ and $k \in K$ such that $aka = k$.
4. There exist numbers c, d such that for any $a \in A_{\mathfrak{p}}^+ = \mathrm{e}^{\mathfrak{a}_{\mathfrak{p}}^+}$, $\Phi(a) \leq c\,\mathrm{e}^{-\rho(\ln(a))}(1 + \sigma(a))^d$, where $\mathfrak{a}_{\mathfrak{p}}^+ = \{X_{\mathfrak{p}} : \alpha(X_{\mathfrak{p}}) \geq 0$ for all α in the positive Weyl chamber in $\mathfrak{a}_{\mathfrak{p}}\}$.
5. There exists a number $r \geq 0$ such that $\int_{SU_{p,q}} \Phi^2(x)(1 + \sigma(x))^{-r}\,\mathrm{d}\mu(x) < \infty$.
6. There exists a number $r \geq 0$ such that (see Section 4) $\int_{SU_{p,q}} |D_{n-1}(x)|^{-1/2}\Phi(x)(1 + \sigma(x))^{-r}\,\mathrm{d}\mu(x) < \infty$.

For $f \in C^\infty(SU_{p,q})$, $D \in \mathbf{D}(SU_{p,q})$ and $r \geq 0$, we define a seminorm $\|f\|_{D,r} = \sup_{g \in SU_{p,q}} |Df(g)|(1 + \sigma(g))^r \Phi^{-1}(g)$. Let $\mathcal{S}(SU_{p,q})$ denote the *Schwartz space*, consisting of all functions $f \in C^\infty(SU_{p,q})$ such that $\|f\|_{D,r} < \infty$ for all $D \in \mathbf{D}(SU_{p,q})$ and $r \geq 0$. One usually topologizes $\mathcal{S}(SU_{p,q})$ by means of the set of seminorms $\|f\|_{D,r}$ which make $\mathcal{S}(SU_{p,q})$ a Fréchet space. Clearly, $C_{\mathrm{c}}^\infty(SU_{p,q}) \subseteq \mathcal{S}(SU_{p,q})$ is a continuous inclusion, and $C_{\mathrm{c}}^\infty(SU_{p,q})$ is dense in $\mathcal{S}(SU_{p,q})$. Also, the inclusion $\mathcal{S}(SU_{p,q}) \subset \mathcal{L}^2(SU_{p,q})$ is continuous.

Let $\mathcal{S}_j(SU_{p,q})$, $0 \leq j \leq p$ denote the set of all functions $f \in \mathcal{S}(SU_{p,q})$ such that $\mathcal{S}_j(SU_{p,q})$ is a closed subset of $\mathcal{S}(SU_{p,q})$ for each j, and $\mathcal{S}(SU_{p,q}) = \oplus_j \mathcal{S}_j(SU_{p,q})$, the sum being smooth. Let Π_j denote the projection of $\mathcal{S}(SU_{p,q})$ on $\mathcal{S}_j(SU_{p,q})$ corresponding to the above direct sum. Then by smoothness, Π_j are continuous endomorphisms of $\mathcal{S}(SU_{p,q})$. Let $\mathcal{H}_j$ denote the closure of $\mathcal{S}_j(SU_{p,q})$ in the Hilbert space $\mathcal{H} = \mathcal{L}^2(SU_{p,q})$. Then $\mathcal{H}$ is the orthogonal sum of $\mathcal{H}_j$, $0 \leq j \leq p$, and $\Pi_j f$, $f \in \mathcal{S}(SU_{p,q})$ is actually the orthogonal projection of f in $\mathcal{H}_j$.

Now, taking $\ln$ in (3.7), one finds that the vector part of $\ln(h_{(j)})$ is given by the matrix with $t_k = \Re(z_k)$ at the position given by the $(p - k + 1)$th row and $(p + k)$th column and also at the position given by the $(p + k)$th row and the $(p - k + 1)$th column, and having all other entries zero. Hence if $h_{(j)} \in H_j$ then we have

$$\sigma(h_{(j)}) = \sqrt{2}(t_1^2 + \cdots + t_j^2)^{1/2} \tag{12.1}$$

showing that the mapping $h_{(j)} \mapsto (\sigma(h_{(j)}))^2$ is a quadratic form on $SU_{p,q}$. As defined generally in [10–12], a distribution Θ on $SU_{p,q}$ is called tempered if it admits a unique continuous extension to the Schwartz space $\mathcal{S}(SU_{p,q})$. It has been proved that an invariant and $\mathfrak{Z}$-finite distribution Θ on $SU_{p,q}$ is tempered if and only if we can choose $c, r \geq 0$ such that $|D_{n-1}(g)|^{1/2}|\Theta(g)| \leq c(1+\sigma(g))^r \; \forall g \in G$. In other words, if $\{H_j\}$ denotes a maximal set of mutually non-conjugate Cartan subgroups of $SU_{p,q}$, then the above condition implies that

$$\sup_{h_{(j)} \in H_j} (1+\sigma(h_{(j)}))^{-r} |D_{n-1}(h_{(j)})|^{1/2} |\Theta(h_{(j)})| < \infty. \tag{12.2}$$

Furthermore, if Θ is tempered, then $\Theta(f) = \int_G f(g)\Theta(g)\,\mathrm{d}\mu(g)$ for $f \in \mathcal{S}(SU_{p,q})$.

Theorem 12.1. *The invariant eigendistribution T_{ω_j} determined by*

$$\Xi_{j\Lambda_\varrho}(h_{(j)}) = \epsilon_{(j)}(h_{(j)})\Delta_{H_j}(h_{(j)})T_{\omega_j}(h_{(j)}), \quad h_{(j)} \in H_j,\ 0 \leq j \leq p$$

is tempered if and only if there exists a positive integer r such that for every j

$$\left(1 + \sqrt{2(t_{(j)1}^2 + \cdots + t_{(j)j}^2)}\right)^{-r} |\Xi_{j\Lambda_\varrho}(h_{(j)})|$$

is bounded by some constant, where $t_{(j)k} = \Re(z_{(j)k})$ for $1 \leq k \leq j$.

Proof. Follows from (12.1) and (12.2). □

13. Invariant eigendistributions of contragradient representations

Let $g \mapsto \rho_j(g)$ be an irreducible unitary representation of $SU_{p,q}$ on a Hilbert space $\mathcal{H}_j$ and $\rho_j \in \omega_j \in \mathcal{E}(SU_{p,q})$, where $\mathcal{E}(SU_{p,q})$ denotes the set of all equivalence classes of irreducible unitary representations of $SU_{p,q}$. Let $g^\dagger \in SU_{p,q}$ denote the inverse of transposed g. Let $\mathcal{H}_j^*$ denote the topological dual of $\mathcal{H}_j$, $\mathcal{H}_j^*$ being equipped with the topology of bounded convergence. Let $\mathcal{H}_j^\dagger$ be the subspace of $\mathcal{H}_j^*$ consisting of those $\varphi^* \in \mathcal{H}_j^*$ for which the rule $g \mapsto \rho_j(g^\dagger)\varphi^*$ defines a continuous map of $SU_{p,q}$ into $\mathcal{H}_j^*$. The space $\mathcal{H}_j^\dagger$ is closed in $\mathcal{H}_j^*$, and the mapping $g \mapsto \rho_j(g^\dagger)$ defines the *contragradient representation* of $g \mapsto \rho_j(g)$ on $\mathcal{H}_j^\dagger$. We denote the contragradient representation by $(\rho_j^\dagger, \mathcal{H}_j^\dagger)$ (or simply $\rho_j^\dagger$). If $T_{\rho_j}(g)$ is the character of $g \mapsto \rho_j(g)$, then the character $T_{\rho_j}^\dagger(g)$ of its contragradient representation is given by $T_{\rho_j}^\dagger(g) = T_{\rho_j}(g^\dagger) = T_{\rho_j}(\bar{g})$, where the complex conjugate element $\bar{g}$ of g is conjugate to $g^\dagger$ under some inner automorphism of $SU_{p,q}$.

Theorem 13.1. *Let $T_{\omega_j^0}(\mathrm{e}^{\iota\phi}, \mathrm{e}^{z}, \mathrm{e}^{\iota\psi})$ be the invariant eigendistribution of an irreducible unitary representation $g \mapsto \rho_j(g)$ in the discrete series ω_j^0 of $SU_{p,q}$. Then the corresponding contragradient representation $g \mapsto \rho_j(g^\dagger)$ is also in the discrete series; its invariant* *eigendistribution is given by* $T^{\dagger}_{\omega_j^0}(\mathrm{e}^{-\iota\phi}, \mathrm{e}^{\bar{z}}, \mathrm{e}^{-\iota\psi})$ *and is specified by the integer parameters* $\{-m_{(j)n-1}, -m_{(j)n-2}, \ldots, -m_{(j)1}\}$ *satisfying* $m_{(j)1} > m_{(j)2} > \cdots > m_{(j)p}$ *and* $m_{(j)p+1} > m_{(j)p+2} > \cdots > m_{(j)n-1}$.

Proof. As $h_{(j)} \rightsquigarrow \bar{h}_{(j)}$, where

$$\begin{aligned} h_{(j)} &= \mathrm{diag}\{\mathrm{e}^{\iota\phi_{(j)1}}, \ldots, \mathrm{e}^{\iota\phi_{(j)p-j}}, \mathrm{e}^{z_{(j)j}}, \ldots, \mathrm{e}^{z_{(j)1}}, \mathrm{e}^{-\bar{z}_{(j)1}}, \ldots, \mathrm{e}^{-\bar{z}_{(j)j}},\\ &\quad \mathrm{e}^{\iota\psi_{(j)q-j}}, \ldots, \mathrm{e}^{\iota\psi_{(j)1}}\},\\ \bar{h}_{(j)} &= \mathrm{diag}\{\mathrm{e}^{-\iota\phi_{(j)1}}, \ldots, \mathrm{e}^{-\iota\phi_{(j)p-j}}, \mathrm{e}^{\bar{z}_{(j)j}}, \ldots, \mathrm{e}^{\bar{z}_{(j)1}}, \mathrm{e}^{-z_{(j)1}}, \ldots, \mathrm{e}^{-z_{(j)j}},\\ &\quad \mathrm{e}^{-\iota\psi_{(j)q-j}}, \ldots, \mathrm{e}^{-\iota\psi_{(j)1}}\}. \end{aligned}$$

one conveniently chooses from Section 11 that $I_p \rightsquigarrow \{n-k : k \in I_p\} := I_p^{\dagger}$. The negative sign in each exponent can be absorbed with the parameters $m_{(j)1}, \ldots, m_{(j)n}$ determining the character. A suitable ordering may be obtained from $\hat{I}_n \bar{h}_{(j)} \hat{I}_n$ (see Section 2). From Theorem 4.3 and from (4.9) and (4.10), we have $\Delta_{H_j}(\bar{h}_{(j)}) = (-1)^{(1/2)n(n-1)+j} \Delta_{H_j}(h_{(j)})$ and $\epsilon_{(j)\mathrm{R}}(\bar{h}_{(j)}) = (-1)^j \epsilon_{(j)\mathrm{R}}(h_{(j)})$. Furthermore, from Section 11 one defines the following permutations:

$$[\sigma_p] \rightsquigarrow (-1)^{p(q-1)} \mathrm{sgn} \begin{pmatrix} 1 & \cdots & p & p+1 & \cdots & n-1 \\ n-i_{n-1} & \cdots & n-i_{q+1} & n-i_q & \cdots & n-i_1 \end{pmatrix} := [\sigma_p^{\dagger}],$$

$$\begin{aligned} &[{}_j\sigma_a] \ \ \mathrm{sgn} \begin{pmatrix} n-i_p & \cdots & n-i_{p-a+2} & n-i_{p-a+1} & \cdots & n-i_{p-a-j+2} & n-i_{p-a-j+1} & \cdots & n-i_1 \\ n-i'_{a-1} & \cdots & n-i'_1 & n-i_a & \cdots & n-i_{a+j-1} & n-i'_p & \cdots & n-i'_{a+j} \end{pmatrix}\\ &\quad = (-1)^{j(p-j)+(1/2)j(j-1)}[{}_j\sigma_a] := [\sigma_a^{\dagger}],\\ &[{}_j\sigma_b] \ \ \mathrm{sgn} \begin{pmatrix} n-i_{n-1} & \cdots & n-i_{n-b+1} & n-i_{n-b} & \cdots & n-i_{n-b-j+1} & n-i_{n-b-j} & \cdots & n-i_{p+1} \\ n-i'_{b-1} & \cdots & n-i'_{p+1} & n-i_b & \cdots & n-i_{b+j-1} & n-i'_{n-1} & \cdots & n-i'_{b+j} \end{pmatrix}\\ &\quad = (-1)^{j(q-j-1)+(1/2)j(j-1)}[{}_j\sigma_b] := [\sigma_b^{\dagger}]. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} {}_{I_a}A(\mathrm{e}^{-\iota\phi}) &= \det \begin{pmatrix} (\mathrm{e}^{\iota\phi_{(j)1}})^{-m_{(j)i'_{p-j}}} & \cdots & (\mathrm{e}^{\iota\phi_{(j)1}})^{-m_{(j)i'_1}} \\ \vdots & \vdots & \vdots \\ (\mathrm{e}^{\iota\phi_{(j)p-j}})^{-m_{(j)i'_{p-j}}} & \cdots & (\mathrm{e}^{\iota\phi_{(j)p-j}})^{-m_{(j)i'_1}} \end{pmatrix}\\ &= (-1)^{(1/2)(p-j)(p-j-1)} \det \begin{pmatrix} (\mathrm{e}^{\iota\phi_{(j)1}})^{-m_{(j)i'_1}} & \cdots & (\mathrm{e}^{\iota\phi_{(j)1}})^{-m_{(j)i'_{p-j}}} \\ \vdots & \vdots & \vdots \\ (\mathrm{e}^{\iota\phi_{(j)p-j}})^{-m_{(j)i'_1}} & \cdots & (\mathrm{e}^{\iota\phi_{(j)p-j}})^{-m_{(j)i'_{p-j}}} \end{pmatrix}\\ &= (-1)^{(1/2)(p-j)(p-j-1)}\, {}_{I_a}A(\mathrm{e}^{\iota\phi}) \end{aligned}$$

with

$$-m_{(j)i'_1} > -m_{(j)i'_2} > \cdots > -m_{(j)i'_{p-j}} \quad \text{or} \quad m_{(j)i'_{p-j}} > m_{(j)i'_{p-j-1}} > \cdots > m_{(j)i'_1}.$$

$$\begin{aligned} A_{I_b}(\mathrm{e}^{-\iota\psi}) &= \det\begin{pmatrix} (\mathrm{e}^{\iota\psi_{(j)q-j}})^{-m_{(j)i'_{n-1}}} & \cdots & (\mathrm{e}^{\iota\psi_{(j)q-j}})^{-m_{(j)i'_{p+j+1}}} \\ \vdots & \vdots & \vdots \\ (\mathrm{e}^{\iota\psi_{(j)2}})^{-m_{(j)i'_{n-1}}} & \cdots & (\mathrm{e}^{\iota\psi_{(j)2}})^{-m_{(j)i'_{p+j+1}}} \end{pmatrix} \\ &= (-1)^{(q-j-1)(q-j-2)}\det\begin{pmatrix} (\mathrm{e}^{\iota\psi_{(j)q-j}})^{-m_{(j)i'_{p+j+1}}} & \cdots & (\mathrm{e}^{\iota\psi_{(j)q-j}})^{-m_{(j)i'_{n-1}}} \\ \vdots & \vdots & \vdots \\ (\mathrm{e}^{\iota\psi_{(j)2}})^{-m_{(j)i'_{p+j+1}}} & \cdots & (\mathrm{e}^{\iota\psi_{(j)2}})^{-m_{(j)i'_{n-1}}} \end{pmatrix} \\ &= (-1)^{(1/2)(q-j-1)(q-j-2)} A_{I_b}(\mathrm{e}^{\iota\psi}) \end{aligned}$$

with

$$-m_{(j)i'_{p+j+1}} > -m_{(j)i'_{p+j+2}} > \cdots > -m_{(j)i'_{n-1}} \quad \text{or} \quad m_{(j)i'_{n-1}} > m_{(j)i'_{n-2}} > \cdots > m_{(j)i'_{p+j+1}}.$$

$${}_{I_a}S^0_{I_b}(\mathrm{e}^{\bar{z}}) = (-1)^j\, {}_{I_a}S^0_{I_b}(\mathrm{e}^{z}).$$

Substituting the above, for a given $I_p^\dagger$, in

$$T^\dagger_{\omega^0_j}(\mathrm{e}^{-\iota\phi}, \mathrm{e}^{\bar{z}}, \mathrm{e}^{-\iota\psi}) = \left(\frac{\epsilon_{(j)\mathrm{R}}(\bar{h}_{(j)})}{\Delta_{H_j}(\bar{h}_{(j)})}\right)\sum_{a,b}[\sigma_a^\dagger][\sigma_b^\dagger]_{I_a}A(\mathrm{e}^{-\iota\phi})_{I_a}S^0_{I_b}(\mathrm{e}^{\bar{z}})A_{I_b}(\mathrm{e}^{-\iota\psi}),$$

one obtains the desired result. □

14. Adjoint of invariant eigendistributions

Let $g \mapsto \rho_j(g), g \in SU_{p,q}$ be an irreducible representation of $SU_{p,q}$ on a Hilbert space $\mathcal{H}_j$ so that $\rho_j(g) \in \mathrm{Aut}(\mathcal{H}_j)$. Then, where $\rho_j^*(g^{-1})$ denotes the adjoint operator of $\rho_{(j)}(g^{-1})$, the homomorphism $g \mapsto \rho_j^*(g^{-1})$ is also an irreducible representation of $SU_{p,q}$ on $\mathcal{H}_j$. The representations $g \mapsto \rho_j(g)$ and $g \mapsto \rho_j^*(g^{-1})$ are equivalent if and only if there exists a non-degenerate continuous Hermitian inner product on $\mathcal{H}_j$ which is invariant under $\rho_j(g), g \in SU_{p,q}$. Let T_{ω_j} be the distribution corresponding to the character of the representation $g \mapsto \rho_j(g)$ of $SU_{p,q}$. One defines its adjoint distribution, denoted by $T^*_{\omega_j}$, by

$$\int_{SU_{p,q}} f(x)T^*_{\omega_j}(x)\,\mathrm{d}\mu(x) = \mathrm{conj}\left\{\int_{SU_{p,q}} f^*(x)T_{\omega_j}(x)\,\mathrm{d}\mu(x)\right\},$$

where $f^*(x) = \overline{f(x^{-1})}$, $\mathrm{d}\mu(x)$ is a Haar measure on $SU_{p,q}$, and $\mathrm{conj}\{a\}$ denotes the complex conjugate number $\bar{a}$ of $a \in \mathbb{C}$. A distribution T_{ω_j} on $SU_{p,q}$ is called self-adjoint if $T_{\omega_j} = T^*_{\omega_j}$.

Lemma 14.1. *$T^*_{\omega_j}$, $0 \leq j \leq p$ is an invariant eigendistribution on $SU_{p,q}$.*

Proof. Let us use the notation of Section 10. Let $\mathfrak{z} \in \mathfrak{Z}$, and let $\bar{\mathfrak{z}} \in \mathcal{Z}(\mathbf{D}(SU_{p,q}))$ be the differential operator on $SU_{p,q}$ defined by, for $f \in C^\infty_{\mathrm{c}}(SU_{p,q})$, $(\bar{\mathfrak{z}}f)(x) = \mathrm{conj}\{(\mathfrak{z}\bar{f})(x)\}$, where $\bar{f}(x) = \mathrm{conj}\{f(x)\}$. Clearly, $\bar{\mathfrak{z}} \in \mathfrak{Z}$, $\overline{a\mathfrak{z}} = \bar{a}\bar{\mathfrak{z}}$ for any $a \in \mathbb{C}$, and $\overline{\mathfrak{z}_1\mathfrak{z}_2} = \overline{\mathfrak{z}_1}\,\overline{\mathfrak{z}_2}$ for any $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathfrak{Z}$. Now, for any $f \in C^\infty_{\mathrm{c}}(SU_{p,q})$,

$$(\mathfrak{z}f^*)(x) = \mathfrak{z}(\bar{f}(x^{-1})) = \mathrm{conj}\{\bar{\mathfrak{z}}f(x^{-1})\} = \mathrm{conj}\{(\hat{\bar{\mathfrak{z}}}f)(x^{-1})\} = (\hat{\bar{\mathfrak{z}}}f)^*(x).$$

Therefore, replacing $\mathfrak{z}$ by $\bar{\mathfrak{z}}$ one has $\bar{\mathfrak{z}}(f^*) = (\hat{\mathfrak{z}}f)^*\ \forall \mathfrak{z} \in \mathfrak{Z}$, and from (see Proposition 10.6) $(\mathfrak{z}T_{\omega_j})(f) = \chi_{\omega_j}(\mathfrak{z})T_{\omega_j}(f)$ we get

$$\begin{aligned}(\mathfrak{z}T^*_{\omega_j})(f) &= T^*_{\omega_j}(\hat{\mathfrak{z}}f) = \int_{SU_{p,q}} (\hat{\mathfrak{z}}f)(x)T^*_{\omega_j}(x)\,\mathrm{d}\mu(x)\\ &= \mathrm{conj}\left\{\int_{SU_{p,q}} (\hat{\mathfrak{z}}f)^*(x)T_{\omega_j}(x)\,\mathrm{d}\mu(x)\right\}\\ &= \mathrm{conj}\left\{\int_{SU_{p,q}} (\bar{\mathfrak{z}}f^*)(x)T_{\omega_j}(x)\,\mathrm{d}\mu(x)\right\} = \mathrm{conj}\{T_{\omega_j}(\bar{\mathfrak{z}}f^*)\}\\ &= \mathrm{conj}\{(\hat{\bar{\mathfrak{z}}}T_{\omega_j})(f^*)\} = \mathrm{conj}\{\chi_{\omega_j}(\hat{\bar{\mathfrak{z}}})T_{\omega_j}(f^*)\} = \mathrm{conj}\{\chi_{\omega_j}(\hat{\bar{\mathfrak{z}}})\}\,\mathrm{conj}\{T_{\omega_j}(f^*)\}\\ &= \mathrm{conj}\{\chi_{\omega_j}(\hat{\bar{\mathfrak{z}}})\}\,\mathrm{conj}\left\{\int_{SU_{p,q}} f^*(x)T_{\omega_j}(x)\,\mathrm{d}\mu(x)\right\}\\ &= \mathrm{conj}\{\chi_{\omega_j}(\hat{\bar{\mathfrak{z}}})\}\int_{SU_{p,q}} f(x)T^*_{\omega_j}(x)\,\mathrm{d}\mu(x) = \mathrm{conj}\{\chi_{\omega_j}(\hat{\bar{\mathfrak{z}}})\}T^*_{\omega_j}(f).\end{aligned}$$

Furthermore, $T^*_{\omega_j}$ is clearly invariant. Thus $T^*_{\omega_j}$ is an invariant eigendistribution with the infinitesimal character, denoted by $\chi^*_{\omega_j} \in \mathrm{Hom}(\mathfrak{Z}, \mathbb{C})$, of ρ^*_j, given by $\chi^*_{\omega_j}(\mathfrak{z}) = \mathrm{conj}\{\chi_{\omega_j}(\hat{\bar{\mathfrak{z}}})\}$. □

Lemma 14.2. *$T^*_{\omega_j}(g) = \overline{T_{\omega_j}(g^{-1})}$, $0 \leq j \leq p$, $g \in SU_{p,q}$.*

Proof. By the definition of adjoint distribution one has

$$\begin{aligned}&\int_{SU_{p,q}} f(x)T^*_{\omega_j}(x)\,\mathrm{d}\mu(x)\\ &\quad= \mathrm{conj}\left\{\int_{SU_{p,q}} f^*(x)T_{\omega_j}(x)\,\mathrm{d}\mu(x)\right\} = \mathrm{conj}\left\{\int_{SU_{p,q}} \overline{f(x^{-1})}T_{\omega_j}(x)\,\mathrm{d}\mu(x)\right\}\\ &\quad= \mathrm{conj}\left\{\int_{SU_{p,q}} \overline{f(x)}T_{\omega_j}(x^{-1})\,\mathrm{d}\mu(x)\right\} = \int_{SU_{p,q}} f(x)\,\mathrm{conj}\{T_{\omega_j}(x^{-1})\}\,\mathrm{d}\mu(x).\end{aligned}$$

Hence, one obtains $T^*_{\omega_j}(g) = \mathrm{conj}\{T_{\omega_j}(g^{-1})\}\,\forall g \in SU_{p,q}$. □

Lemma 14.3. $\Xi^*_{j\Lambda_\varrho}(h_{(j)}) = \mathrm{conj}\{\Xi_{j\Lambda_\varrho}(h_{(j)})\}, h_{(j)} \in H_j, 0 \le j \le p$.

Proof. From (10.5) and from the above result, since

$$\mathrm{conj}\{\epsilon_{(j)\mathrm{R}}(h^{-1}_{(j)})\Delta_{H_j}(h^{-1}_{(j)})\} = \epsilon_{(j)\mathrm{R}}(h_{(j)})\Delta_{H_j}(h_{(j)}),$$

one has immediately that $\Xi^*_{j\Lambda_\varrho}(h_{(j)}) = \mathrm{conj}\{\Xi_{j\Lambda_\varrho}(h^{-1}_{(j)})\}\,\forall h_{(j)} \in H_j$. □

Proposition 14.4. *T_{ω_j} is self-adjoint if and only if $T^*_{\omega_j}(g) = \mathrm{conj}\{T_{\omega_j}(g^{-1})\}, 0 \le j \le p, g \in SU_{p,q}$, i.e., if and only if $\Xi_{j\Lambda_\varrho}(h_{(j)}) = \mathrm{conj}\{\Xi_{j\Lambda_\varrho}(h^{-1}_{(j)})\}, h'_{(j)} \in H_j$.*

Proof. The proof follows from the definition of self-adjoint distribution and from Lemmas 14.2 and 14.3. □

Proposition 14.5. *If $u'_{(j)} = (u'_{(j)1}, \ldots, u'_{(j)n-1})$ corresponds to the infinitesimal character* χ_{ω_j} of T_{ω_j}, *then that of $T^*_{\omega_j}$ is given by $\overline{u'}_j = (\overline{u'}_{(j)1}, \ldots, \overline{u'}_{(j)n-1})$, and, for some $w \in W(\mathfrak{s}l_n(\mathbb{C}), \mathfrak{h}^{\mathrm{c}}_j) \cong S_n$, one has that $\overline{u'}_j = w(u'_j)$.*

Proof. Follows immediately if one takes the complex conjugation operation on Eqs. (8.2) and (10.8). □

An infinitesimal character χ_{ω_j} (or $u'_{(j)}$) is self-adjoint if $\chi^*_{\omega_j}(\mathfrak{z}) = \chi_{\omega_j}(\mathfrak{z})\,\forall \mathfrak{z} \in \mathfrak{Z}$, or equivalently, if $\mathcal{X}(u'_{(j)}) = \mathcal{X}(u'_{(j)})\,\forall \mathcal{X} \in \mathfrak{S}(\mathfrak{s}l_n(\mathbb{C}))$. Hence from Proposition 14.5, $u'_{(j)}$ is self-adjoint if and only if $\overline{u'}_{(j)} = w(u'_{(j)})$ for some $w \in S_n$.

Let $\mathfrak{T}(u'_{(j)})$ denote the set of all invariant eigendistributions on $SU_{p,q}$ and let $\mathfrak{T}_\mathrm{s}(u'_{(j)})$ denote its subset consisting of all self-adjoint invariant eigendistributions. Analogously, let $\mathfrak{K}(u'_{(j)})$ be the set of all functions $\Xi_{j\Lambda_\varrho}$ (which are analytic on H_j) satisfying Eq. (10.8), and $\mathfrak{K}_\mathrm{s}(u'_{(j)})$ be its subset consisting of all $\Xi_{j\Lambda_\varrho} \in \mathfrak{K}(u'_{(j)})$ functions satisfying the condition $\Xi^*_{j\Lambda_\varrho}(h_{(j)}) = \Xi_{j\Lambda_\varrho}(h_{(j)})$. By Lemma 14.2 the adjoint $T^*_{\omega_j}$ is an element of $\mathfrak{T}(u'_{(j)})$. Hence $T'_{\omega_j} = \frac{1}{2}(T_{\omega_j} + T^*_{\omega_j})$ and $T''_{\omega_j} = (1/2\sqrt{-1})(T_{\omega_j} - T^*_{\omega_j})$ are in $\mathfrak{T}_\mathrm{s}(u'_{(j)})$. Thus we have the following theorem.

Theorem 14.6. *Assume that $u'_{(j)}$ is self-adjoint. Then any distribution $T_{\omega_j} \in \mathfrak{T}(u'_{(j)})$ is expressed uniquely as $T_{\omega_j} = T'_{\omega_j} + \sqrt{-1}T''_{\omega_j}$, where $T'_{\omega_j}, T''_{\omega_j} \in \mathfrak{T}_\mathrm{s}(u'_{(j)})$. Analogously, any function $\Xi_{j\Lambda_\varrho} \in \mathfrak{K}(u'_{(j)})$ is uniquely expressed as $\Xi_{j\Lambda_\varrho} = \Xi'_{j\Lambda_\varrho} + \sqrt{-1}\Xi''_{j\Lambda_\varrho}$, where $\Xi'_{j\Lambda_\varrho}, \Xi''_{j\Lambda_\varrho} \in \mathfrak{K}_\mathrm{s}(u'_{(j)})$.*

Remark. In the representation theory of Lie algebras one comes across unbounded operators. The Hellinger–Toeplitz theorem implies that unbounded self-adjoint operators cannot be defined in all of the Hilbert space $\mathcal{H}$. One therefore has to associate with every unbounded operator $\mathcal{O}$ its domain of definition $D_\mathcal{O}$ (e.g., the Gårding domain). If $D_{\mathcal{O}^*}$ is dense in $\mathcal{H}$, then one defines $\mathcal{O}^{**}$, called the self-adjoint extension of $\mathcal{O}$. If $\mathcal{O}^{**} = \mathcal{O}^{*}$ then $\mathcal{O}$ is said to be essentially self-adjoint. Following the Nelson–Stinespring theorem [29] one can show that for every elliptic element $\mathfrak{X}$ (as elliptic differential operator) of the enveloping algebra $\mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}))$, the (algebraic) representation operator $T(\mathfrak{X})$ is essentially self-adjoint.

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